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INSTITUTE OF MATHEMATICS

ÚSTAV MATEMATIKY

USING GENERALIZED FUNCTIONS IN CONTINUUM MECHANICS

VYUŽITÍ ZOBECNĚNÝCH FUNKCÍ V MECHANICE KONTINUA

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

AUTHOR Bc. Petr Procházka

AUTOR PRÁCE

SUPERVISOR prof. Ing. František Pochylý, CSc.

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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Using generalized functions in continuum mechanics

Concise characteristic of the task:

Basically the work will build on previous bachelor thesis, which dealt with problems of statics. This thesis will extend the issue of the time–dependent problems of solid and fluid mechanics. The aim is to establish a mathematical model of above mentioned tasks, using the generalized functions.

Goals Master's Thesis:

1. Create a mathematical model of a beam vibrations. The beam has n–flexible supports. Solution will be assumed on the generalized functions principle.
2. Create software for the solution of the own and forced beam vibrations.
3. Consider the use of generalized functions for hydrodynamic purposes.

Recommended bibliography:

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SUMMARY

This thesis deals with the utilization of distributions or generalized functions in solving non-stationary boundary problems in continuum mechanics. At first, the theory of distributions and their definition as continuous linear functionals on the test function space is introduced. The second part of the theoretical framework presents Laplace integral transform. The following chapter deals with the beam deflections under the discontinuous time variable loads. It results in the creation of a general model of the deflection lines using the distributions. The last chapter deals with the solution of non-stationary flow in pipes connected by various hydraulic elements.

KEYWORDS

Generalized functions, distributions, beam deflection, non-stationary flow in pipe, piping systems

ABSTRAKT

Tato práce se zabývá využitím distribucí neboli zobecněných funkcí k řešení nestacionárních okrajových problémů v mechanice kontinua. Nejprve je zavedena teorie distribucí a jejich definice jako spojitých lineárních funkcionálů na prostoru testovacích funkcí. Druhá část teoretické kapitoly představuje Laplaceovu integrální transformaci. Následující kapitola se věnuje řešení průhybu nosníků pod vlivem nespojitého časově proměnlivého zatížení. Jejím výsledkem je vytvoření obecného modelu řešení průhybových čar nosníků využitím distribucí. Poslední kapitola se zabývá řešením nestacionárního proudění v trubicích spojených hydraulickými prvky.

KLÍČOVÁ SLOVA

Zobecněné funkce, distribuce, průhybová čára nosníku, nestacionární proudění v trubici, potrubní systémy

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ROZŠÍŘENÝ ABSTRAKT

V mechanice kontinua se často řeší parciální diferenciální rovnice čtyř proměnných, kde první proměnnou je čas a zbylé tři jsou prostorové. Tyto rovnice jsou doplněny o počáteční a okrajové podmínky.

V dnešní době vysokého výpočetního výkonu a specializovaných softwarů je možné získat velmi přesné řešení takto definovaných problémů během několika minut. Tyto výsledky je možné velmi dobře vizualizovat a interpretovat, stále jsou však pouze numerickými a často nenabízejí možnost hlubšího porozumění problému. Při návrhu nových konceptů či optimalizaci stávajících může být užitečná znalost analytického řešení, jehož získání může být velmi náročné. V takových případech se hledají vhodné matematické metody, které přehlednými a časově ne příliš náročnými přitom ale dostatečně exaktními prostředky, umožňují najít požadované řešení.

Jednou z takových metod je i známá metoda integrální transformace vícenásobná *Laplaceova transformace*, která převádí parciální diferenciální rovnice na rovnice diferenciální. Tato transformace je popsána ve druhé části teoretické kapitoly a navazuje na teorii distribucí, neboli zobecněných funkcí, která je nástrojem umožňujícím řešit některé problémy mechaniky kontinua.

Za zakladatele teorie distribucí je považován francouzský matematik Laurent Schwartz, který za ni v roce 1950 dostal Fieldsovu medaili[1]. Schwartz teorii distribucí vystavěl na spojitých lineárních funkcionálech definovaných na prostoru testovacích funkcí. Distribuce zobecňují pojem funkce, a lze tedy každou lokálně integrovatelnou funkci reprezentovat distribucí. Takto definované funkcionály se nazývají *regulární distribuce.*

Tato práce se zaměřuje na využití druhé skupiny distribucí, a to *singulárních*, které nelze zadat pomocí lokálně integrovatelných funkcí. Takovou zobecněnou funkcí je *Diracova delta funkce*, která umožňuje matematickou interpretaci nespojitého zatížení nosníků a potrubí v dalších oddílech této práce.

Druhá kapitola se věnuje využití distribucí při výpočtech průhybové čáry nosníku uloženého na pružném podloží, pro kterou je odvozena parciální diferenciální rovnice čtvrtého řádu podle modelu nosníku Euler-Bernoulli

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{q_v}{EI} - \frac{m_v'}{EI}. \tag{2.13}$$

Právě *Diracova delta funkce* je vhodným nástrojem k implementaci bodových nestacionárních zatížení, kterými jsou síly, momenty a pružiny. *Laplaceova transformace* převede uvedený problém na snáze řešitelný, jak je ukázáno v části 2.2. Následuje analýza uvažovaného vetknutého nosníku s volným koncem, ve které jsou určeny vlastní frekvence pomocí analytického přístupu. V programu ANSYS byly určeny vlastní frekvence téhož nosníku pomocí numerické simulace a Tabulka 2.1 ukazuje jejich srovnání. Vyšší hodnoty vlastních frekvencí určených analyticky jsou způsobeny zjednodušením v uvažovaném modelu.

Vliv různých hodnot uvažovaného vnějšího tlumení nosníku je patrný na Obrázku 2.7. Při zatížení nosníku s harmonickým průběhem je žádoucí vyhnout se rezonanci, která je popsána v odstavci 2.2.3. Nosník uvedený v příkladu 2.3 se věnuje možnostem modelování různých typů uložení konce nosníku pomocí pružin. Tabulka 2.8 srovnává přesnost

[1] `https://www.mathunion.org/fileadmin/IMU/Prizes/Fields/1950/index.html`

řešení vetknutého nosníku a dvou modelů s různými tuhostmi pružin. Dále je uvedena simulace přechodu vetknutého nosníku s volným koncem k situaci, kdy je nosník na druhém konci prostě podepřen.

Tyto poznatky jsou využity k vytvoření obecného modelu nosníku uloženého na n-pružinách bodově zatíženého nestacionárními silami a momenty v sekci 2.4 a slouží jako základ algoritmu v přiloženém programu.

Závěrečná kapitola se zabývá prouděním tekutiny v potrubním systému, které je popsáno rovnicemi odvozenými v sekci 3.1
rovnicí kontinuity

$$\frac{S}{\rho a^2}\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{3.47}$$

a *rovnicí rovnováhy*

$$\frac{\rho}{S}\frac{\partial Q}{\partial t} + \frac{\partial p}{\partial x} + \frac{b}{S}Q = 0. \tag{3.48}$$

V první části sekce 3.2 je uveden standardní postup řešení pro jednoduchou přímou trubici. Dvojitá *Laplaceova transformace* $\mathscr{L}\{\cdot\}_{x,t\to\varepsilon,s}$ a vhodné úpravy převedou původní problém na nalezení *Laplaceova originálu* k

$$\mathbf{u}(x,s) = \mathbf{P}(x,s)\cdot\mathbf{u}(0,s), \tag{3.66}$$

kde je nutné vyjádřit vektor okrajových podmínek $\mathbf{u}(0,s)$ na levé straně trubice. Rozšířením tohoto postupu je přechod na potrubní systém, který je tvořen sériově zapojenými jednoduchými trubicemi spojenými *hydraulickými prvky* z Tabulky 3.1. Takový systém se řeší pomocí matic přechodu $\mathbf{R}$ jednotlivých *hydraulických prvků*, které definují okrajové podmínky pro levý konec trubice a umožňují využít vzorec (3.66) pro jednotlivé trubice systému.

Diracova delta funkce vyjádří bodové zatížení *hydraulickými prvky* v případě řešení pomocí distribucí. V takovém případě řešíme soustavu rovnic

$$\frac{\partial p(x,t)}{\partial x} + \frac{\rho}{S}\frac{\partial Q(x,t)}{\partial t} - \sigma\, p(\zeta,t)\,\delta(x-\zeta) + b\, Q(\zeta,t)\,\delta(x-\zeta) = 0 \tag{3.87}$$

$$\frac{\partial Q(x,t)}{\partial x} + \frac{S}{\rho a^2}\frac{\partial p(x,t)}{\partial t} + C\frac{\partial p(\zeta,t)}{\partial t}\,\delta(x-\zeta) + \kappa\,\frac{\partial Q(\zeta,t)}{\partial t}\,\delta(x-\zeta) = 0, \tag{3.88}$$

kde konstanty σ, b, C, κ charakterizují jednotlivé *hydraulické prvky.* V závěru práce jsou uvedeny způsoby výpočtu oběma metodami pro všechny hydraulické prvky z tabulky 3.1.

Výsledky této práce by se daly rozšířit o vícerozměrné problémy v kapitole 2, případně o rozvětvené potrubní systémy v kapitole 3.

DECLARATION

I declare that I have written the Master's Thesis titled "Using generalized functions in continuum mechanics" independently, under the guidance of the advisor and using exclusively the technical references and other sources of information cited in the thesis and listed in the comprehensive bibliography at the end of the thesis.

Brno . . . .

author's signature

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CONTENTS

INTRODUCTION

In the field of continuum mechanics, problems are often assigned using partial differential equations of four variables, where the first one refers to time and the remaining correspond to the spatial variables. These equations are complemented with initial and boundary conditions.

Today's high-performance computing power and well-optimised specialised industry-specific software make it possible to obtain very precise solutions to these problems within minutes. These results can be easily visualised and interpreted, but they are still numerical and often do not give a proper understanding of the problem. When designing new concepts or optimising existing ones, the notion of an analytical solution may be useful, but the acquisition may be very challenging. In such cases, mathematical methods are explored to find a way how to obtain the solution that is accessible and clear, but not that demanding and time-consuming.

Such an analytical method is the well-known method of integral transformation, the *Laplace transform* which very elegantly converts partial differential equations into differential ones. If there exists a solution to a given task determined by *Laplace transform*, entirely numerical task remains to find the roots of the characteristic equation.

Another tool to solve some of the problems in continuum mechanics is the so-called theory of generalized functions, or distributions, based on continuous linear functionals defined on the space of the test functions. A well-known singular distribution is the so-called *Dirac delta function*, which, as the name suggests (in Latin *singularis* – alone, unique, single, isolated), allows to express the effects of point loads. By using the integral transformation mentioned above, it is possible to solve some problems of the continuum mechanics with such defined point loads.

This thesis builds on the previous bachelor's thesis [10], where the stationary problems of the deflection line of the beam were solved. The last chapter of the mentioned bachelor's thesis showed the possibilities of utilization of the distributions in the hydromechanics, namely of the examples of pressure losses in the direct piping system.

Therefore this work aims to extend the considered cases to non-stationary load behaviour, namely

- Create a mathematical model of beam vibrations. The beam has n-flexible supports. The solution will be assumed on the generalized functions principle.
- Create software for the solution of the natural and forced beam vibrations.
- Consider the use of generalized functions for hydrodynamic purposes.

1 THEORETICAL BASIS

1.1 Theory of distributions

This section covers all the essential tools to define the theory of the generalized functions (distributions) properly. The primary source of this chapter were the books [3], [7] and [12].

1.1.1 Test function space

A domain is any connected open subset $\Omega \subset \mathbb{R}^n$. Differentiability class $C^\infty(\Omega)$ represents all smooth[1] functions defined on Ω. Another concept that plays a crucial role in the theory of distributions is the compact[2] support[3] of a function. Since we are working with functions in $\mathbb{R}^n$, they have compact support if and only if they have bounded support, since the support is closed, by the definition.

Definition 1. The space $\mathscr{D}(\Omega)$ of test functions defined on Ω is the set of all the functions of the class C^∞ with compact supports[4]. The elements $\varphi \in \mathscr{D}(\Omega)$ are called the test functions.

Example 1.1.1. The well-known sample of the test function space $\mathscr{D}(\mathbb{R})$ is defined as follows

$$\varphi(x) = \begin{cases} \exp\left(-\dfrac{a^2}{a-x^2}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1. \end{cases} \tag{1.1}$$

The graph of this function is in the Figure 1.1.

Remark. It is easy to show that the space $\mathscr{D}(\Omega)$, where $\Omega \subset \mathbb{R}^n$, is a vector subspace of the real function space over $\mathbb{R}^n$:

$$\forall \varphi_1(x), \varphi_2(x) \in \mathscr{D}(\Omega) \Rightarrow \varphi_1(x) + \varphi_2(x) \in \mathscr{D}(\Omega),$$
$$\forall \varphi(x) \in \mathscr{D}(\Omega), \forall \lambda \in \mathbb{R}^n \Rightarrow \lambda\varphi(x) \in \mathscr{D}(\Omega).$$

Note. Before we continue, we introduce the concept of so-called *multi-index* in $\mathbb{R}^n$. Let α be an $n-$ tuple of non-negative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define the differential operator D^α as

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},$$

where

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \qquad x^\alpha = x_1^{\alpha_1}\, x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$
$$\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n! \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\alpha!\,(\alpha-\beta)!}.$$

[1]The function has derivatives of all orders.

[2]Set A is compact in $\mathbb{R}^n \iff A$ is closed and bounded.

[3]Support of a function φ is the closure in $\mathbb{R}^n$ of the set of points x where φ is non-zero, i.e.

$$\operatorname{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n | \varphi(x) \neq 0\}}.$$

[4]Functions of the class C^∞ with compact support are usually denoted as C_0^∞.

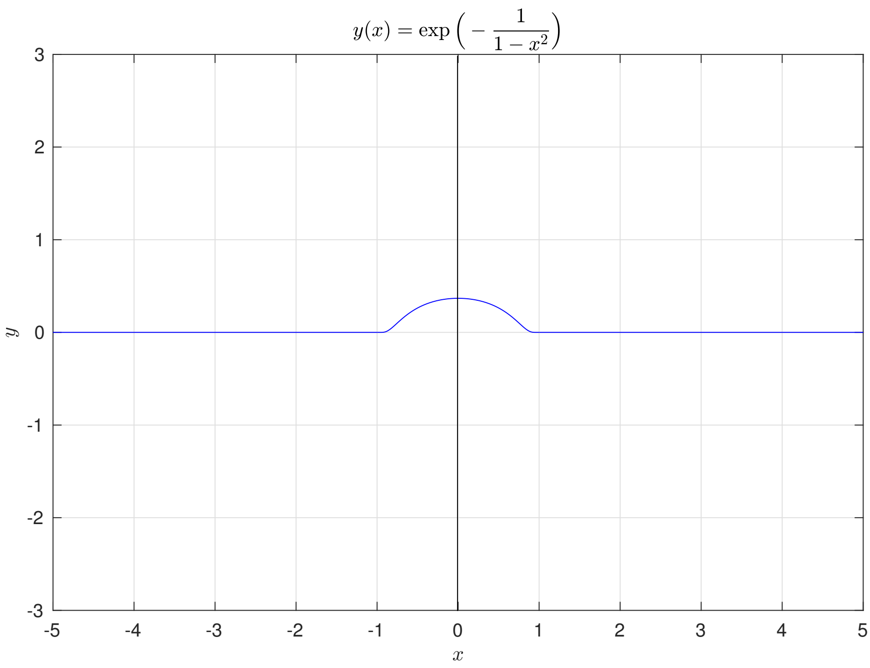


Fig. 1.1: The test function defined by equation (1.1) for $a = 1$.

e.g. in $\mathbb{R}^5$, with $\alpha = (2, 2, 0, 3, 2)$, we get

$$\mathrm{D}^\alpha = \frac{\partial^9}{\partial x_1^2\, \partial x_2^2\, \partial x_4^3\, \partial x_5^2}.$$

It is more convenient to work with the fundamental notion of convergence. Thus we introduce the kind of convergence, denoted as $\xrightarrow{\mathscr{D}}$, which is useful in this case.

Theorem 1. *Sequence φ_n converges to φ in $\mathscr{D}$, if following conditions are satisfied*

- *$\exists\, K$ compact in Ω, that contains all the supports of all φ_n, i.e.* $\mathrm{supp}(\varphi_n) \subset K\, \forall n$,
- *$\mathrm{D}^\alpha \varphi_n \rightrightarrows \mathrm{D}^\alpha \varphi$ in $\Omega\, \forall \alpha$, i.e. all the derivatives of φ_n converge uniformly[5] to the derivatives of φ.*

1.1.2 Distributions

As stated in [2]

Definition 2. A scalar-valued linear map from a linear space $X \to \mathbb{R}$ is called a *linear functional*, or *linear form* on X. The space of linear functionals on X is called the *algebraic dual space of* X, denoted as X^*. And the space of continuous linear functionals on X is called the *topological dual space* of X, denoted as X'.

Definition 3. A continuous linear functional on the space $\mathscr{D}(\Omega)$ of test functions is called *generalized function* or *distribution*, and the value on the test function $\varphi \in \mathscr{D}(\Omega)$ is denoted as $\langle T, \varphi \rangle$.

[5] A sequence of functions $\{f_n\}$ is said to be uniformly convergent to f for values x of a set A, if $\forall \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N$ and $\forall x \in A$.

Therefore the functional $T : \mathscr{D}(\Omega) \to \mathbb{R}$ is the *distribution* if it is

- linear, i.e. $\langle T, c_1\varphi_1 + c_2\varphi_2\rangle = c_1\langle T, \varphi_1\rangle + c_2\langle T, \varphi_2\rangle \;\; \forall \varphi_i \in \mathscr{D}(\Omega),\ c_i \in \mathbb{R}$,
- and continous, i.e. $\varphi_n \xrightarrow{\mathscr{D}} \varphi \Rightarrow \langle T, \varphi_n\rangle \to \langle T, \varphi\rangle \, \forall \varphi \in \mathscr{D}(\Omega)$

Note. Because the distribution T is linear, it is enough to require

$$\varphi_n \xrightarrow{\mathscr{D}} 0 \Rightarrow \langle T, \varphi_n\rangle \to 0.$$

Remark. We shall use the following notation for the integral (in the Lebesgue sense) of a function $f(x) = f(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$

$$\int_R \int_R \cdots \int_R f(x_1, x_2, \ldots, x_n) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n = \int_{R^n} f(x) \, \mathrm{d}x,$$

where $\mathrm{d}x = \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n$.

Definition 4. A function $f(x)$ is locally integrable in Ω, written as $f \in L^1_{loc}(\Omega)$, if it is a Lebesgue measurable function and $\int_K |f(x)| \, \mathrm{d}x < +\infty$ for all compact subsets $K \subset \Omega$.

Distributions are called generalized functions; thus they should contain the standard functions. Locally integrable functions generate such a set of distributions. For example, every locally integrable function $f(x)$ in Ω determines a distribution T_f by the formula

$$\langle T_f, \varphi\rangle = \int_\Omega f(x) \, \varphi(x) \, \mathrm{d}x, \;\; \varphi \in \mathscr{D}(\Omega). \tag{1.2}$$

Because this distribution is obviously linear, we only show the continuity. Due to the fact that the function $\varphi \in \mathscr{D}(\Omega)$ it is bounded, i.e. $|\varphi(x)| \le k$, and that it has a compact support K, the integral in (1.2) is finite

$$\left| \int_\Omega f(x) \, \varphi(x) \, \mathrm{d}x \right| = \left| \int_K f(x) \, \varphi(x) \, \mathrm{d}x \right| \le k \int_K |f(x)| \, \mathrm{d}x < +\infty.$$

Since the sequence $\varphi_n \to 0$, then so does $\langle T_f, \varphi_n\rangle$. Hence it is continuous; it defines a distribution.

Remark. The distributions, which meet (1.2) are called *regular distributions.*

Theorem 2. *Two functions f, g define the same functional $T_f = T_g$, if and only if they are equal almost everywhere.*

Note. Proof of this theorem is in the book [12] at the page 75.

Let us take two locally integrable functions f, g, which are equal almost everywhere, so we will be talking only about the classes of locally integrable functions. Now we can consider distributions as the generalization of locally integrable functions as we identify such functions, defined almost everywhere, with functionals T_f

$$\langle f, \varphi\rangle = \langle T_f, \varphi\rangle = \int_\Omega f(x) \, \varphi(x) \, \mathrm{d}x.$$

Example 1.1.2. Regular distribution defined by the so-called Heaviside function[6]

$$\mathrm{H}(x-a) = \mathrm{H}_a(x) = \langle \mathrm{H}_a, \varphi\rangle = \int_a^\infty \varphi(x) \mathrm{d}x.$$

[6] Heaviside function in $\mathbb{R}$ is defined as $\mathrm{H}(x-a) = \mathrm{H}_a(x) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x \ge a \end{cases}.$

Since φ has a compact support and $\mathrm{H}(x-a)$ is a piecewise continuous function, this is a regular distribution.

There is a group of continuous linear functionals defined on the test function space $\mathscr{D}(\Omega)$ which do not define a distribution by the formula (1.2). Those generalized functions are known as *singular distributions*.

Example 1.1.3. The well-known singular distribution is the *Dirac delta function*, defined as

$$\delta(x-a) = \delta_a = \langle \delta_a, \varphi \rangle = \varphi(a). \tag{1.3}$$

1.1.3 Operations with distributions

According to the Definition 2, we denote the space of all continuous linear functionals defined on space $\mathscr{D}(\Omega)$ as $\mathscr{D}'(\Omega)$, which is a linear space. This *topological dual space* of $\mathscr{D}(\Omega)$ is larger than $\mathscr{D}(\Omega)$ itself, as it contains functions as the *Dirac delta*. Many operations which are defined for smooth functions with compact support are defined for distributions as well.

Definition 5. For distributions T, T_1, $T_2 \in \mathscr{D}'(\Omega)$, where $\Omega \subset \mathbb{R}$, we define the basic operations as follows

- Sum of $T_1 + T_2$

$$\langle T_1 + T_2, \varphi \rangle \stackrel{def}{=} \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle, \ \forall \varphi \in \mathscr{D}(\Omega),$$

- Scalar product αT, for all $\alpha \in \mathbb{R}$

$$\langle \alpha T, \varphi \rangle \stackrel{def}{=} \alpha \langle T, \varphi \rangle, \ \forall \varphi \in \mathscr{D}(\Omega),$$

- Multiplication by a smooth function gT, for all $g \in C^{\infty}(\Omega)$

$$\langle gT, \varphi \rangle \stackrel{def}{=} \langle T, g\varphi \rangle, \ \forall \varphi \in \mathscr{D}(\Omega).$$

Differentiation of a distribution

In the beginning, we derive the formula for differentiation in $\mathbb{R}$ and after that, we expand it to $\mathbb{R}^n$.

We use so-called *integration by parts* formula for the integrable functions f and φ over the line segment $I = (a, b)$

$$\int_I f'(x)\, \varphi(x) \, \mathrm{d}x = [f(x)\, \varphi(x)]_b^a - \int_I f(x)\, \varphi'(x) \, \mathrm{d}x. \tag{1.4}$$

If φ has its compact support in $I = (a, b)$, then $\varphi(a) = \varphi(b) = 0$. Hence the minuend in (1.4) is cancelled. The same situation is if the interval $I = (-\infty, \infty)$ is unbounded, because the limits

$$\lim_{x \to \pm\infty} \varphi(x) = 0.$$

We use this fact to define the derivatives of the distributions, so instead of differentiating the distribution, we will differentiate the test function, which is C_0^{∞}. Therefore we get the formulas for the first and in general for the kth derivative.

$$\langle T', \varphi \rangle \stackrel{def}{=} -\langle T, \varphi' \rangle,$$
$$\langle T^{(k)}, \varphi \rangle \stackrel{def}{=} (-1)^k \langle T, \varphi^{(k)} \rangle.$$

In the linear spaces with higher dimensions, the formula for distribution's derivatives will be derived from the *Green's identity*.

$$\int_K \frac{\partial f(x)}{\partial x_i}\, \varphi(x)\,\mathrm{d}x = \int_{\partial K} f(x)\,\varphi(x)\,n_i\,\mathrm{d}S - \int_K f(x) \frac{\partial \varphi(x)}{\partial x_i}\,\mathrm{d}x. \tag{1.5}$$

This identity is valid for differentiable functions with smooth boundary. Due to the fact that φ with a compact support K is zero-valued near the border ∂K, the minuend in (1.5) is cancelled,thus we can write the general definition for the derivatives of distributions.

Definition 6. The α-derivative $\partial^\alpha T$ of the distribution T is defined as

$$\langle \partial^\alpha T, \varphi \rangle \stackrel{def}{=} (-1)^{|\alpha|} \langle T, \mathrm{D}^\alpha \varphi \rangle. \tag{1.6}$$

Example 1.1.4. Derivative of the *Heaviside distribution*, defined in Example 1.1.2, where $\Omega \subset \mathbb{R}$ and $\alpha = (\alpha_1) = (1)$

$$\begin{aligned}\langle \partial^\alpha \mathrm{H}_a, \varphi \rangle = -\langle \mathrm{H}_a, \mathrm{D}^\alpha \varphi \rangle = -\langle \mathrm{H}_a, \varphi' \rangle = -\int_{-\infty}^{+\infty} \mathrm{H}_a\, \varphi'(x)\,\mathrm{d}x = -\int_a^{+\infty} \varphi'(x)\,\mathrm{d}x \\ = -[\varphi(x)]_a^{+\infty} = -(0 - \varphi(a)) = \langle \delta_a, \varphi \rangle.\end{aligned}$$

1.2 *Laplace transform*

Laplace transform is widely used integral transform, which was discovered and named after Pierre-Simon Laplace. This method elegantly reduces linear differential equations down to their more easily handled analogues (i.e. ODEs to algebraic equations and PDEs to ODEs), which makes the analysis much more viable.

In this section, we briefly define this transform and give important results and theorems for solving boundary problems introduced later in this work. More details and features can be found in [6], [8] and [11].

1.2.1 Definition of the *Laplace transform*

Let $f(t)$ be (in general) a complex function of real variable t defined on the interval $[t_0, \infty)$ and absolutely summable for $t_0 \le t \le t_e$, i.e. the integral

$$\int_{t_0}^{t_e} |f(t)|\,\mathrm{d}t$$

converges. Additionally we require that $f : [t_0, \infty) \to \mathbb{C}$ ($\mathbb{R}$) be of an *exponential order*[7]. Let the function $f(t)$ have finitely many points of discontinuity on a finite interval

[7]We say, that function f is of an exponential order if there exist constants $A > 0, B \in \mathbb{R}$ such that

$$|f(t)| \le A\,\mathrm{e}^{tB}\,\forall t \ge t_0.$$

and $f(t) = 0, \forall t < t_0$. Considering all the previous requirements and setting $t_0 = 0$ we can define the *Laplace transform* integral[8]

$$\mathscr{L}\{f(t)\}_{t\to s} = \int_{-\infty}^{+\infty} f(t)\,\mathrm{e}^{-st}\mathrm{d}t = \int_{0}^{+\infty} f(t)\,\mathrm{e}^{-st}\mathrm{d}t = F(s).$$

$F(s)$ is called a *Laplace transform* of a function $f(t)$.

Let $T(x)$ be a distribution, which satisfy all the requirements in the definition of the *Laplace transform.* Then we can define the *Laplace transform* for distribution $T(x)$ as follows

$$\mathscr{L}\{T\}_{x\to\varepsilon} \overset{def}{=} \langle T, \mathrm{e}^{-\varepsilon x}\rangle. \tag{1.7}$$

Example 1.2.1. *Laplace transform* of the Dirac distribution δ_a and its first derivative

$$\mathscr{L}\{\delta_a\}_{x\to\varepsilon} = F(\varepsilon) = \langle \delta_a, \mathrm{e}^{-\varepsilon x}\rangle = \mathrm{e}^{-\varepsilon a},$$
$$\mathscr{L}\{\delta_a'\}_{x\to\varepsilon} = F(\varepsilon) = \langle \delta_a', \mathrm{e}^{-\varepsilon x}\rangle = -\langle \delta_a, (\mathrm{e}^{-\varepsilon x})'\rangle = -(-\varepsilon)\mathrm{e}^{-\varepsilon a} = \varepsilon\mathrm{e}^{-\varepsilon a}.$$

1.2.2 Basic theorems of the *Laplace transform*

Here are some essential theorems and features that will be used in the following chapters. Proofs of all these statements can be found in [11].

Linearity property

From the *Laplace transform* definition integral it follows that for arbitrary constants $c_1, c_2 \in \mathbb{C}(\mathbb{R})$

$$\mathscr{L}\{c_1 f(t) + c_2 g(t)\}_{t\to s} = c_1\,\mathscr{L}\{f(t)\}_{t\to s} + c_2\,\mathscr{L}\{g(t)\}_{t\to s}.$$

Derivative theorem

Let $f(t)$ be continuous for $t > 0$ up to the order n, then

$$\mathscr{L}\{f^{(n)}(t)\}_{t\to s} = s^n\mathscr{L}\{f(t)\}_{t\to s} - s^{n-1}\,f(0+) - s^{n-2}\,f'(0+) - \ldots - s\,f^{(n-2)}(0+) - f^{(n-1)}(0+),$$

where $f^{(i)}(0+)$ stands for $\lim_{t\to 0+} f^{(i)}(t)$.

Theorem of derivative by the parameter λ

Let $f(t,\lambda)$ be the function of variable t and parameter λ. Its *Laplace transform* according to the parameter t is $\mathscr{L}\{f(t,\lambda)\}_{t\to s} = F(s,\lambda)$ and if $\frac{\partial}{\partial\lambda} f(t,\lambda)$ exists $\forall t > 0$, then

$$\mathscr{L}\left\{\frac{\partial}{\partial\lambda} f(t,\lambda)\right\}_{t\to s} = \frac{\partial}{\partial\lambda} F(s,\lambda).$$

First shifting theorem

Let $\mathscr{L}\{f(t)\}_{t\to s} = F(s)$ and $a \in \mathbb{C}\,(\mathbb{R})$, such that $\operatorname{Re} s > \operatorname{Re} a$, then

$$\mathscr{L}\{f(t)\,\mathrm{e}^{at}\}_{t\to s} = F(s-a).$$

[8] In case that $t_0 \neq 0$, we call this transform *generalized Laplace transform.*

Second shifting theorem

Let $f(t)$ be *Laplace transformable* function and $\mathrm{H}(t)$ a Heaviside function, if $a \in \mathbb{C}\,(\mathbb{R})$, then

$$\mathscr{L}\{f(t-a)\mathrm{H}(t-a)\}_{t\to s} = \mathrm{e}^{-as}\mathscr{L}\{f(t)\} = \mathrm{e}^{-as}F(s),$$

and its inverse form

$$\mathscr{L}^{-1}\{\mathrm{e}^{-as}F(s)\}_{t\to s} = f(t-a)\,\mathrm{H}(t-a).$$

Convolution theorem

Let $F(s) = \mathscr{L}\{f(t)\}_{t\to s}$ and $G(s) = \mathscr{L}\{g(t)\}_{t\to s}$, then

$$\mathscr{L}\{f(t)*g(t)\}_{t\to s} = F(s)\cdot G(s) = \mathscr{L}\bigg\{\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau\bigg\}_{t\to s} = \mathscr{L}\bigg\{\int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau\bigg\}_{t\to s},$$

where the integral $\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau$ is called the convolution integral and the convolution of functions $f(t)$ and $g(t)$ is denoted as $f(t) * g(t)$.

Determining the original to the *Laplace transform*

The task to find the original function to the *Laplace transform* image is usually the harder part. The complete definition with proofs can be found in [6] in chapter 8.2. We state here so-called *complex inversion formula.*

Theorem 3. *Let the meromorphic function $F(s) = \frac{A(s)}{B(s)}$ have a countable number of poles s_k of order n_k, $k = 1, 2, \dots$ and $\sum_k n_k = n$. With the exceptions of points at the poles let there hold that*

$$\lim_{|s|\to\infty} F(s) = 0.$$

Then the decomposition of $F(s)$ into partial fractions is

$$F(s) = \sum_k \left(\frac{A_{k_1}}{s-s_k} + \frac{A_{k_2}}{(s-s_k)^2} + \ldots + \frac{A_{k_{n_k}}}{(s-s_k)^{n_k}}\right), \tag{1.8}$$

thus the original to the image $F(s)$ turns out to be

$$f(t) = \sum_k \left(A_{k_1}\,\mathrm{e}^{s_k t} + A_{k_2}\frac{t}{1!}\,\mathrm{e}^{s_k t} + \ldots + A_{k_{n_k}}\frac{t^{n_k-1}}{(n_k-1)!}\,\mathrm{e}^{s_k t}\right).$$

The coefficients A_{kl} are determined by multiplying the total decomposition (1.8) *by the factor $(s-s_k)^{n_k}$. It removes all the poles s_k and after that the modified equation* (1.8) *is differentiated $(n_k - l)$ times, what gives us* [9]

$$\frac{\mathrm{d}^{n_k-l}}{\mathrm{d}s^{n_k-l}}\left(F(s)\cdot(s-s_k)^{n_k}\right) = \ldots + \left(\frac{(n_k-1)!}{(l-1)!}\,A_{k_1}\,(s-s_k)^{l-1} + \frac{(n_k-2)!}{(l-2)!}\,A_{k_2}\,(s-s_k)^{l-2} + \ldots + (n_k-l)!\,A_{k_l}\right) + \ldots$$

[9] We write down only the terms which belong to the pole s_k.

If we limit $s \to s_k$, all the terms of RHS in the previous equation are cancelled except the absolute term $(n_k - l)!\, A_{k_l}$, hence

$$A_{k_l} = \frac{1}{(n_k - l)!} \lim_{s \to s_k} \frac{\mathrm{d}^{n_k - l}}{\mathrm{d}\, s^{n_k - l}} \Big(F(s)\,(s - s_k)^{n_k} \Big).$$

Example 1.2.2. Let the poles of $B(s)$ be simple, then the partial fractions decomposition of $F(s) = \frac{A(s)}{B(s)}$ is

$$F(s) = \sum_k \Big(\frac{A_{k_1}}{s - s_k} \Big),$$

where

$$A_{k_1} = \frac{1}{0!} \lim_{s \to s_k} \Big(\frac{\mathrm{d}}{\mathrm{d}s} \Big)^0 \Big(\frac{A(s)(s - s_k)}{B(s)} \Big) = \lim_{s \to s_k} \left(\frac{A(s)}{\frac{B(s) - B(s_k)}{s - s_k}} \right) = \frac{A(s_k)}{B'(s_k)}.$$

And we are able to easily write down the original to this *Laplace image* as

$$f(t) = \sum_k \frac{A(s_k)}{B'(s_k)}\, \mathrm{e}^{s_k t}.$$

1.2.3 Multidimensional *Laplace transform*

In the previous subsection, the definition and underlying theory of the one-dimensional *Laplace transform* was given. Mentioned integral transformation reduces the problems given by ordinary differential equations to algebraic ones. This part of the thesis introduces a method that will provide a simplification of the problems assigned by partial differential equations.

In the linear partial differential equation, each independent variable can be transformed entirely independently of the transformation by any other independent variable. Therefore it is advantageous to find a solution of a linear partial differential equation with constant coefficients using *Laplace transform* according to each independent variable.

Note. The multidimensional[10] *Laplace transform* will be denoted as

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty f(x, y, \ldots, z)\, \mathrm{e}^{-\varepsilon x - q y - \ldots - r z} \mathrm{d}x\, \mathrm{d}y \ldots \mathrm{d}z = \mathscr{L}\{f(x, y, \ldots, z)\}_{x, y, \ldots, z \to \varepsilon, q, \ldots r},$$

i.e. in our case in the following chapters

$$\mathscr{L}\{f(x,t)\}_{x,t \to \varepsilon, s} = \int_0^\infty \int_0^\infty f(x,t)\, \mathrm{e}^{-\varepsilon x - s t} \mathrm{d}x\, \mathrm{d}t = F(\varepsilon, s).$$

Note. We denote

$$f_{x^k}(a,t) = \frac{\partial^k}{\partial x^k} f(x,t) \Big|_{x=a} \text{ and similarly } f_{t^l}(x,b) = \frac{\partial^l}{\partial t^l} f(x,t) \Big|_{t=b}$$

Let $f(x,t)$ be continuous and *Laplace* transformable for $0 \le x < \infty$ and $0 \le t < \infty$, then we have

$$\mathscr{L}\Big\{ \frac{\partial^m}{\partial x^m} f(x,t) \Big\}_{x,t \to \varepsilon, s} = \varepsilon^m \{f(x,t)\}_{x,t \to \varepsilon, s} - \varepsilon^{m-1} \{f(0,t)\}_{t \to s} - \varepsilon^{m-2} \{f_x(0,t)\}_{t \to s} - \ldots - \varepsilon \{f_{x^{m-2}}(0,t)\}_{t \to s} - \{f_{x^{m-2}}(0,t)\}_{t \to s},$$

[10] Therefore we understand the importance of writing down the subscript in the transformation.

similarly

$$\mathscr{L}\left\{\frac{\partial^n}{\partial t^n}f(x,t)\right\}_{x,t\to\varepsilon,s} = s^n\{f(x,t)\}_{x,t\to\varepsilon,s} - s^{n-1}\{f(x,0)\}_{x\to\varepsilon} - s^{n-2}\{f_t(x,0)\}_{x\to\varepsilon} - \\ \ldots - s\{f_{t^{n-2}}(x,0)\}_{x\to\varepsilon} - \{f_{t^{n-2}}(x,0)\}_{x\to\varepsilon},$$

and finally $\mathscr{L}\left\{\frac{\partial^{m+n}}{\partial x^m\,\partial t^n}f(x,t)\right\}_{x,t\to\varepsilon,s} =$

$$\begin{array}{c|r}
\varepsilon^m s^n\{f(x,t)\}_{x,t\to\varepsilon,s} & -\varepsilon^{m-1}s^n\{f(0,t)\}_{t\to s} - \varepsilon^{m-2}s^n\{f_x(0,t)\}_{t\to s} - \\
 & \ldots - s^n\{f_{x^{m-1}}(0,t)\}_{t\to s} \\
\hline
-\varepsilon^m s^{n-1}\{f(x,0)\}_{x\to\varepsilon} & +\varepsilon^{m-1}s^{n-1}f(0,0) + \varepsilon^{m-2}s^{n-1}f_x(0,0) + \\
 & \ldots + s^{n-1}f_{x^{m-1}}(0,0) \\
-\varepsilon^m s^{n-2}\{f_t(x,0)\}_{x\to\varepsilon} & +\varepsilon^{m-1}s^{n-2}f_t(0,0) + \varepsilon^{m-2}s^{n-2}f_{xt}(0,0) + \\
 & \ldots + s^{n-2}f_{x^{m-1}t}(0,0) \\
\vdots & \vdots \\
-\varepsilon^m\{f_{t^{n-1}}(x,0)\}_{x\to\varepsilon} & +\varepsilon^{m-1}f_{t^{n-1}}(0,0) + \varepsilon^{m-2}f_{xt^{n-1}}(0,0) + \\
 & \ldots + f_{x^{m-1}t^{n-1}}(0,0).
\end{array}$$

1.3 Supplementary mathematical instruments

Functions and operators, which will help us in the following chapters are defined in this section. These tools will provide more comfortable and shorter notation.

1.3.1 Rayleigh functions

In the computational part of this thesis, namely Chapter 2 and Chapter 3, we will often use combinations of sin, cos, sinh and cosh, therefore it seems convenient to introduce the so-called *Rayleigh functions*[11], which are defined as

$$S(\lambda x) \overset{def}{=} \frac{1}{2}(\cosh(\lambda x) + \cos(\lambda x)) \qquad T(\lambda x) \overset{def}{=} \frac{1}{2}(\sinh(\lambda x) + \sin(\lambda x))$$
$$U(\lambda x) \overset{def}{=} \frac{1}{2}(\cosh(\lambda x) - \cos(\lambda x)) \qquad V(\lambda x) \overset{def}{=} \frac{1}{2}(\sinh(\lambda x) - \sin(\lambda x)).$$

thus their derivatives

$$\begin{array}{ccc}
S' = \lambda V & S'' = \lambda^2 U & S''' = \lambda^3 T \\
T' = \lambda S & T'' = \lambda^2 V & T''' = \lambda^3 U \\
U' = \lambda T & U'' = \lambda^2 S & U''' = \lambda^3 V \\
V' = \lambda U & V'' = \lambda^2 T & V''' = \lambda^3 S,
\end{array}$$

[11] In Czech they are known as *Krylovy funkce*.

The *Laplace transform* and power series expansion of the *Rayleigh functions* are

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{\varepsilon^4-\lambda^4}\right\}_{x\to\varepsilon} &= \frac{1}{2\lambda^3}\left(\sinh(\lambda x)-\sin(\lambda x)\right) = \frac{V(\lambda x)}{\lambda^3} = \left(\frac{x^3}{3!}+\frac{\lambda^4x^7}{7!}+\frac{\lambda^8x^{11}}{11!}+\dots\right)\\
\mathscr{L}^{-1}\left\{\frac{\varepsilon}{\varepsilon^4-\lambda^4}\right\}_{x\to\varepsilon} &= \frac{1}{2\lambda^2}\left(\cosh(\lambda x)-\cos(\lambda x)\right) = \frac{U(\lambda x)}{\lambda^2} = \left(\frac{x^2}{2}+\frac{\lambda^4x^6}{6!}+\frac{\lambda^8x^{10}}{10!}+\dots\right)\\
\mathscr{L}^{-1}\left\{\frac{\varepsilon^2}{\varepsilon^4-\lambda^4}\right\}_{x\to\varepsilon} &= \frac{1}{2\lambda}\left(\sinh(\lambda x)+\sin(\lambda x)\right) = \frac{T(\lambda x)}{\lambda} = \left(x+\frac{\lambda^4x^5}{5!}+\frac{\lambda^8x^9}{9!}+\dots\right)\\
\mathscr{L}^{-1}\left\{\frac{\varepsilon^3}{\varepsilon^4-\lambda^4}\right\}_{x\to\varepsilon} &= \frac{1}{2}\left(\cosh(\lambda x)+\cos(\lambda x)\right) = S(\lambda x) = \left(1+\frac{\lambda^4x^4}{4!}+\frac{\lambda^8x^8}{8!}+\dots\right).
\end{aligned}
\tag{1.9}
$$

1.3.2 Multi-point differential operator

While applying the integral transformation to multi-point or boundary value problem, it is convenient to introduce a new symbol by the following definition.

Definition 7. The symbols $\mathcal{E}_a^j$, S_a^j, etc. that multiply the *Laplace transform* from the left in the variables ε or s, respectively, are numerical operators that give the image a numerical value of the $j-$th derivative of its original at point a (assuming it exists), e.g. for the *Laplace transform* $\mathscr{L}\{f(x)\}_{x\to\varepsilon}=F(\varepsilon)$, it applies as $\mathcal{E}_a^jF(\varepsilon)=\mathcal{E}_a^j\{f(x)\}_{x\to\varepsilon}=f^{(j)}(a)$.

The choice of the letter of this numeric operator is always selected by a capital letter, the lower case of which denotes the variable of the given *Laplace transform.* For images of multiple *Laplace transforms*, we have

$$
\begin{aligned}
\mathcal{E}_a\{f(x,t)\}_{x,t\to\varepsilon,s} &= \{f(a,t)\}_{t\to s},\\
S_b^3\{f(x,t)\}_{x,t\to\varepsilon,s} &= \left\{\frac{\partial^3}{\partial t^3}f(x,t)\bigg|_{t=b}\right\}_{x\to\varepsilon}.
\end{aligned}
$$

Numeric operators relating to the same variable are often preferred to assemble in a numeric operator vector, e.g.

$$
\mathcal{E}=\begin{bmatrix}\mathcal{E}_a^2 & \mathcal{E}_b^5 & \cdots & \mathcal{E}_c^j\end{bmatrix}^{\mathrm{T}},\ \mathcal{E}_a=\begin{bmatrix}\mathcal{E}_a^4 & \mathcal{E}_a^6 & \cdots & \mathcal{E}_a^j\end{bmatrix}^{\mathrm{T}},
$$

which are applied as

$$
\mathcal{E}\{f(x)\}_{x\to\varepsilon}=\begin{bmatrix}f''(a)\\ f^{(5)}(b)\\ \vdots\\ f^{(j)}(c)\end{bmatrix},\ \mathcal{E}_a\{f(x)\}_{x\to\varepsilon}=\begin{bmatrix}f^{(4)}(a)\\ f^{(6)}(a)\\ \vdots\\ f^{(j)}(a)\end{bmatrix}.
$$

2 TRANSVERSE VIBRATIONS OF CONTINUOUS BEAMS

The Euler-Bernoulli, also know as classical beam theory dates back to the 18th century[1]. Named after Jacob Bernoulli, who discovered that the curvature of an elastic beam at a given point is proportional to the bending moment at that point. This theory is simple and provides reasonable engineering approximations, but unfortunately, it tends to overvalue the natural frequencies a little bit. This trend reveals with higher modes, as it is shown in the Table 2.1.

2.1 Derivation of the beam equation

The derivation of the dynamic beam equation is based on the literature [4] and [5].

The beam is a solid body of the length L with small cross-sectional dimensions loaded in a plane containing its centre axis in a direction perpendicular to this axis by a continuous distributed load, dependent generally on the position x and the time t. We assume that the ends of the beam perform the prescribed planar motion in the plane mentioned above. Due to the fact, that the inertial oscillation effects are planar, it is also to be assumed that the central axis of inertia of the cross-section of the beam lies in the previously described plane.

The beam material is characterized by the *Young's modulus* $E(x)$ and density per unit length $\rho(x)$. Both of these variables may be position dependent, but we will consider a homogeneous prismatic beam so they will be constant for the whole beam, as well as a cross section $A(x) = A$ and moment of inertia $I(x) = I$.

According to the Bernoulli-Navier hypothesis, we assume that

- Plane cuts perpendicular to the main axis stay planes and perpendicular to the deflection line even after deformation.
- We can neglect the longitudinal deflections, and cross-sectional skews caused by shear.
- The transverse deflections are assumed to be small so that the deflection line is almost flat and hence the angles of rotation are small.

The vibration of the beam may occur due to the described external load, or by the deflection and simultaneous rotation of the end sections of the beam according to the specified point position function at the initial time. The movement of each beam cross-section is then a general planar motion in the plane described above which can be decomposed for a point on the axis in an upward movement in a direction perpendicular to the axis of the beam described by transverse deflection $y(x, t)$. Also, secondary rotation around an axis perpendicular to the described plane of movement is described by the angle of rotation $\beta(x, t)$.

Suppose that an elementary piece of the length $\mathrm{d}x$ is cut out from the beam at the position x, as we can see in the Figure 2.1.

We have the element in the deformed position, described by transverse deflection $y(x, t)$ and the angle of rotation $\beta(x, t)$ of this cut from the plane YZ. This angle is the tangent angle to the centerline of the element with the x-axis. Due to the small vertical deflections,

[1] According to the [4].

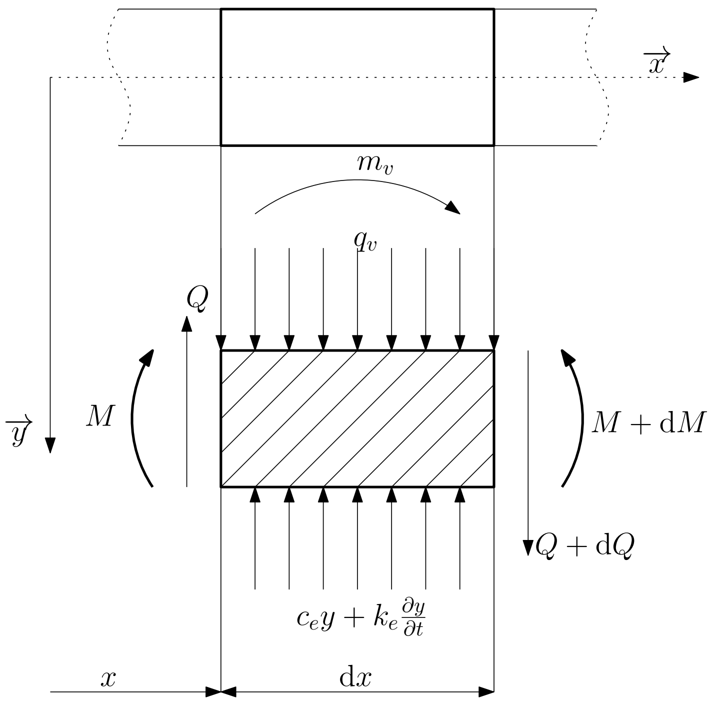


Fig. 2.1: Forces and bending moments diagram for the element dx.

we can write

$$\beta \cong \mathrm{tg}\beta = \frac{\partial y}{\partial x}. \tag{2.1}$$

The inertial force of the element dx is

$$\rho A \frac{\partial^2 y}{\partial t^2} \mathrm{d}x, \tag{2.2}$$

where ρA is actually a mass per unit and we can write the rotational moment of inertia for this element as

$$\rho I \frac{\partial^2 \beta}{\partial t^2} \mathrm{d}x, \tag{2.3}$$

where I is the *area moment of inertia*, to an axis perpendicular to the drawing, which is considered to be a plane of the oscillatory motion.

We will study the effect of the external damping to the beam, which is assumed to be directly proportional to the speed of movement. The damping force against movement per element dx is then given by constant of the external damping k_e

$$k_e \frac{\partial y}{\partial t} \mathrm{d}x. \tag{2.4}$$

According to the Figure 2.1, the equilibrium of the forces acting on the element in the vertical direction is

$$Q + \mathrm{d}Q + q_v \mathrm{d}x - Q - \rho A \frac{\partial^2 y}{\partial t^2} \mathrm{d}x - k_e \frac{\partial y}{\partial t} \mathrm{d}x - c_e y \, \mathrm{d}x = 0,$$

where $\mathrm{d}Q = \frac{\partial Q}{\partial x} \mathrm{d}x$, c_e is the coefficient of the flexible subsoil, q_v is the external loading per unit length and by the adjustments

$$-\frac{\partial Q}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} + k_e \frac{\partial y}{\partial t} + c_e y = q_v. \tag{2.5}$$

The equilibrium of the moments of the forces acting on the element will be given to the centre of gravity of the element dx

$$Q\frac{\mathrm{d}x}{2} + (Q + \mathrm{d}Q)\frac{\mathrm{d}x}{2} + M - (M + \mathrm{d}M) - \rho I \mathrm{d}x\frac{\partial^2 \beta}{\partial t^2} + m_v \mathrm{d}x = 0,$$

where $\mathrm{d}M = \frac{\partial M}{\partial x}\mathrm{d}x$, m_v is the external moment to the unit length and the differentially small expression of second order $\frac{\mathrm{d}Q\,\mathrm{d}x}{2}$ is negligible. And again we adjust the equation into

$$-Q + \frac{\partial M}{\partial x} + \rho I\frac{\partial^2 \beta}{\partial t^2} = m_v. \tag{2.6}$$

The relationship between the bending moment and the exerted variations in the rotation of the cross-sectional plane is

$$M = -EI\frac{\partial \beta}{\partial x}. \tag{2.7}$$

Thus we are left with four equations of four variables Q, y, M, β depending on x, t, from which we will exclude all of them except y. Derivative of equation (2.7) gives

$$\frac{\partial M}{\partial x} = -EI\frac{\partial^2 \beta}{\partial x^2}. \tag{2.8}$$

Fit the expression (2.8) into (2.6)

$$-Q + -EI\frac{\partial^2 \beta}{\partial x^2} + \rho I\frac{\partial^2 \beta}{\partial t^2} = m_v. \tag{2.9}$$

Derivative of (2.9) with respect to x, as we substitute for β in $\frac{\partial^2 \beta}{\partial x^2}$ and $\frac{\partial^2 \beta}{\partial^2 t}$ from (2.1), gives

$$-\frac{\partial Q}{\partial x} = \frac{\partial m_v(x,t)}{\partial x} - \rho I\frac{\partial^4 y}{\partial x^2 \partial t^2} + EI\frac{\partial^4 y}{\partial x^4} \tag{2.10}$$

Substituting (2.10) into (2.5) gives

$$\frac{\partial m_v}{\partial x} - \rho I\frac{\partial^4 y}{\partial x^2 \partial t^2} + EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} + k_e\frac{\partial y}{\partial t} + c_e y = q_v. \tag{2.11}$$

In this work we neglect the effect of rotational inertia, thus with these assumptions we get

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} + k_e\frac{\partial y}{\partial t} + c_e y = q_v - \frac{\partial m_v}{\partial x}, \tag{2.12}$$

which we rewrite into the final form as

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{q_v}{EI} - \frac{m_v'}{EI}. \tag{2.13}$$

2.2 First beam example

Before we define a general mathematical model of a beam on n-flexible supports, we will solve two problems to show the method, procedures and ideas behind the approach using distributions.

In the first case, we have a beam fixed on the *LHS* and free on the other end. After the computation part, the basic characteristics of this particular beam will be determined and shown.

The second beam has a modification such that the *LHS* end is not fixed, but is loaded with rotational and translational springs. We show, how the first example can be modelled by choosing different stiffness of the springs.

And towards the end the general model will be deduced.

2.2.1 General equation

In the first example, we have a beam on a flexible subsoil of length L and parameters: cross section A, density per unit length ρ, *Young's modulus* E and the *area moment of inertia* I. We will consider external damping k_e and the constant of elastic foundation c_e. The left end of the beam is fixed and the right one, which is free, is point loaded with general time-dependent force $F(t)$. This case is illustrated in the Figure 2.2.

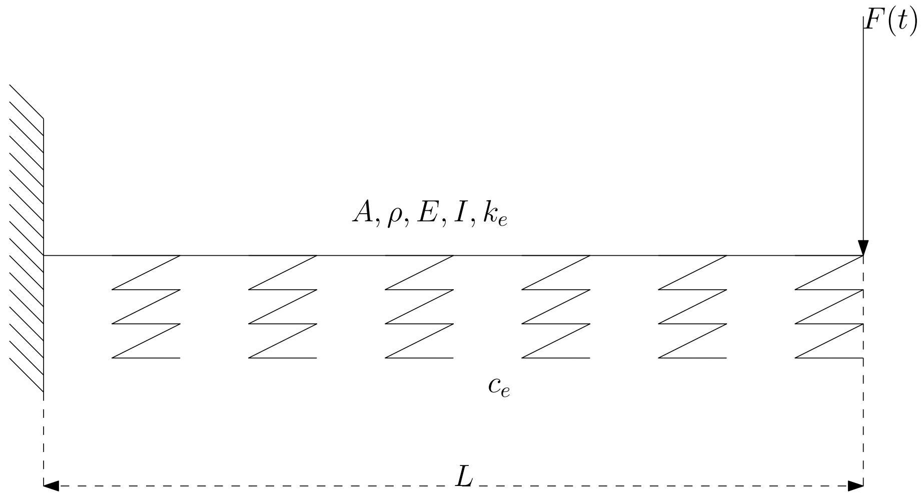


Fig. 2.2: Situation in the beam Example 2 in 2.2

For this problem we have the equation (2.13) in the form

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{F(t)\delta(x-L)}{EI}, \tag{2.14}$$

with boundary conditions

$$y(0,t) = 0 \qquad\qquad y'(0,t) = 0 \tag{2.15}$$

$$y''(L,t) = 0 \qquad\qquad y'''(L,t) = 0, \tag{2.16}$$

and initial conditions

$$y(x,0) = 0 \qquad\qquad \dot{y}(x,0) = 0. \tag{2.17}$$

Laplace transform applied to the (2.14), using boundary conditions (2.15) and initial conditions (2.17) leads us to

$$\left(\varepsilon^4 + \frac{\rho A s^2 + k_e s + c_e}{EI}\right)\{y\}_{x,t\to\varepsilon,s} = \begin{bmatrix}\varepsilon & 1\end{bmatrix}\mathbf{Y} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L}, \tag{2.18}$$

where $\mathbf{Y} = \left\{\begin{bmatrix} y''(0,t) \\ y'''(0,t)\end{bmatrix}\right\}_{t\to s}$. Using the modified *Rayleigh functions*, defined in the previous chapter in (1.9), and denoting $\lambda^4 = -\frac{\rho A s^2 + k_e s + c_e}{EI}$ we get

$$\{y\}_{x,t\to\varepsilon,s} = \frac{1}{\varepsilon^4 - \lambda^4}\left(\begin{bmatrix}\varepsilon & 1\end{bmatrix}\mathbf{Y} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L}\right) = \left\{\begin{bmatrix}\frac{U(\lambda x)}{\lambda^2} & \frac{V(\lambda x)}{\lambda^3}\end{bmatrix}\right\}_{x\to\varepsilon}\mathbf{Y} + \left\{\frac{V\big(\lambda(x-L)\big)}{\lambda^3}\mathrm{H}(x-L)\right\}_{x\to\varepsilon}\frac{\{F(t)\}_{t\to s}}{EI}. \tag{2.19}$$

Applying the boundary condition (2.16) for the right end of the beam with respect to the Definition 7 gives

$$\begin{bmatrix}\mathcal{E}_L^2 \\ \mathcal{E}_L^3\end{bmatrix}\{y\}_{x,t\to\varepsilon,s} = \begin{bmatrix}0 \\ 0\end{bmatrix} = \diamondsuit\mathbf{Y} + \begin{bmatrix}\frac{T\big(\lambda(L-L)\big)}{\lambda} \\ S\big(\lambda(L-L)\big)\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}, \tag{2.20}$$

where

$$\diamondsuit = \begin{bmatrix} S(\lambda L) & \frac{T(\lambda L)}{\lambda} \\ \lambda V(\lambda L) & S(\lambda L)\end{bmatrix}.$$

To express $\mathbf{Y}$, we have to multiply the equation (2.20) by an inverse of a matrix $\diamondsuit$

$$\mathbf{Y} = -\diamondsuit^{-1}\begin{bmatrix}\frac{T\big(\lambda(L-L)\big)}{\lambda} \\ S\big(\lambda(L-L)\big)\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI} = -\diamondsuit^{-1}\begin{bmatrix}0 \\ 1\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}, \tag{2.21}$$

as we denoted $\diamondsuit^{-1} = \frac{1}{\det(\diamondsuit)}\mathrm{adj}(\diamondsuit)$ the inverse of $\diamondsuit$. Substituting (2.21) into (2.19) we get a solution that suits all the boundary conditions

$$\{y\}_{x,t\to\varepsilon,s} = -\left\{\begin{bmatrix}\frac{U(\lambda x)}{\lambda^2} & \frac{V(\lambda x)}{\lambda^3}\end{bmatrix}\right\}_{x\to\varepsilon}\diamondsuit^{-1}\begin{bmatrix}0 \\ 1\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI} + \mathrm{H}(x-L)\left\{\frac{V(\lambda(x-L))}{\lambda^3}\right\}_{x\to\varepsilon}\frac{\{F(t)\}_{t\to s}}{EI}. \tag{2.22}$$

Since we only consider x belonging to the beam, $0 \le x \le L$, the previous equation reduces to

$$\{y\}_{x,t\to\varepsilon,s} = -\left\{\begin{bmatrix}\frac{U(\lambda x)}{\lambda^2} & \frac{V(\lambda x)}{\lambda^3}\end{bmatrix}\right\}_{x\to\varepsilon}\diamondsuit^{-1}\begin{bmatrix}0 \\ 1\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}. \tag{2.23}$$

To identify the original function in variable t, we have to decompose the meromorphic function of the parameter λ^4 in the partial fractions according to the poles that determine the roots of the characteristic equation. It is possible to show, that for this function, the regular part will drop in partial fraction decomposition, as a result of having no significant singularities in infinity.

$$\det\Big(\diamondsuit\Big) = S^2(\lambda L) - T(\lambda L)V(\lambda L) = (\cosh(\lambda L)\cos(\lambda L) + 1). \tag{2.24}$$

Denoting the roots as λ_j, for $j = 1, 2, \ldots$, and using the Theorem 3, we decompose the equation (2.23) into the partial fractions and get

$$\begin{aligned}\{y\}_{x,t\to\varepsilon,s} = -\sum_j \left\{\begin{bmatrix}\frac{U(\lambda_j x)}{\lambda_j^2} & \frac{V(\lambda_j x)}{\lambda_j^3}\end{bmatrix}\right\}_{x\to\varepsilon} \begin{bmatrix} S(\lambda_j L) & -\frac{T(\lambda_j L)}{\lambda_j} \\ -\lambda_j V(\lambda_j L) & S(\lambda_j L)\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}. \\ \frac{\{F(t)\}_{t\to s}}{EI}\frac{4}{L}\frac{1}{\frac{S(\lambda_j L)V(\lambda_j L) - T(\lambda_j L)U(\lambda_j L)}{\lambda_j^3}}\frac{1}{\lambda^4-\lambda_j^4}.\end{aligned} \tag{2.25}$$

Substituting back for the expression λ^4 gives

$$\frac{1}{\lambda^4 - \lambda_j^4} = -\frac{1}{\lambda_j^4 - \lambda^4} = -\frac{1}{\lambda_j^4 + \frac{\rho A s^2 + k_e s + c_e}{EI}} = -\frac{EI}{\lambda_j^4\, EI + \rho A s^2 + k_e s + c_e}, \tag{2.26}$$

thus for $-\frac{1}{EI(\lambda^4 - \lambda_j^4)}$ in (2.25)

$$-\frac{1}{EI}\frac{-EI}{\lambda_j^4\, EI + \rho A s^2 + k_e s + c_e} = \frac{1}{\lambda_j^4\, EI + \rho A s^2 + k_e s + c_e}. \tag{2.27}$$

Known *Laplace transform* for $\mathrm{e}^{-\upsilon t}\sin(\Omega\, t)$

$$\mathscr{L}\{\mathrm{e}^{-\upsilon t}\sin(\Omega\, t)\}_{t\to s} = \frac{\Omega}{(s+\upsilon)^2 + \Omega^2},$$

allows (2.27) to express as

$$\frac{1}{\lambda_j^4\, EI + \rho A s^2 + k_e s + c_e} = \frac{1}{\rho A\, \Omega_j}\{\mathrm{e}^{-\upsilon t}\sin(\Omega_j\, t)\}_{t\to s}, \tag{2.28}$$

where $\Omega_j^2 = \dfrac{1}{\rho A}\left(c_e + \lambda_j^4 EI - \dfrac{k_e^2}{4\rho A}\right)$ and $\upsilon = \dfrac{k_e}{2\rho A}$.

Finally we are able to write down the original function with the convolution integral for the external point load $F(t)$ acting at the right end of the beam

$$\begin{aligned} y(x,t) = \sum_j \begin{bmatrix}\frac{U(\lambda_j x)}{\lambda_j^2} & \frac{V(\lambda_j x)}{\lambda_j^3}\end{bmatrix} \begin{bmatrix} S(\lambda_j L) & -\frac{T(\lambda_j L)}{\lambda_j} \\ -\lambda_j V(\lambda_j L) & S(\lambda_j L)\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}. \\ \frac{4}{L}\frac{1}{\frac{S(\lambda_j L)V(\lambda_j L) - T(\lambda_j L)U(\lambda_j L)}{\lambda_j^3}}\frac{1}{\rho A\, \Omega_j}\int_0^t \mathrm{e}^{-\upsilon(t-\tau)}\sin\Big(\Omega_j\,(t-\tau)\Big)F(\tau)\,\mathrm{d}\tau \end{aligned} \tag{2.29}$$

2.2.2 Natural Frequencies and mode shapes

Consider a beam with parameters

- Rectangular cross-section:
  - height of the cross-section: $h = 0.05\,\mathrm{m}$
  - width of the cross-section: $b = 0.3\,\mathrm{m}$
- Length of the beam: $L = 2.5\,\mathrm{m}$
- Density of the beam material: $\rho = 7830\,\mathrm{kg\,m^{-3}}$
- Young's modulus of the beam: $E = 2.05 \times 10^{11}$ Pa

For this example, we will omit the external damping and consider that there is no elastic subsoil. From the previous chapter, we can easily obtain a formula for the natural frequencies

$$\Omega_j = \sqrt{\frac{\lambda_j^4\,EI}{\rho A}}\,\mathrm{rad\,s^{-1}},$$

where λ_j are the imaginary parts of the roots of the characteristic equation (2.24).

The Figure 2.3 shows first five mode shapes of this beam, which were determined from the equation (2.29).

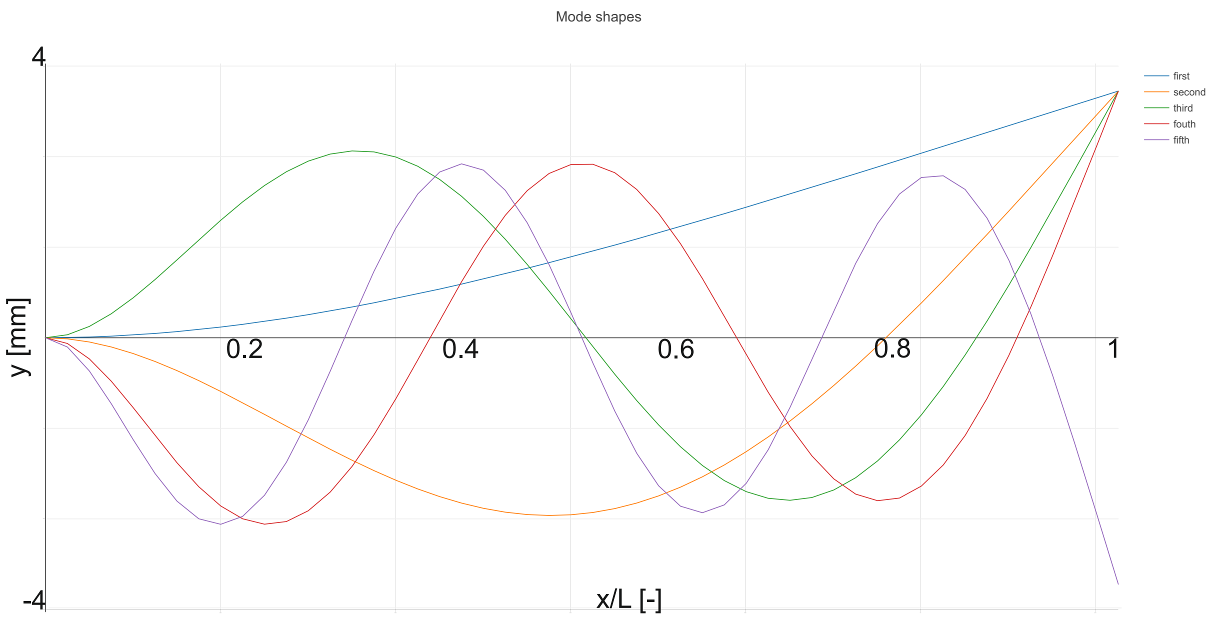


Fig. 2.3: First five mode shapes of the beam in Example 1.

In the Table 2.1 the first six natural frequencies are set out. In the second and third columns are the calculated values, and in the last one, there are the values determined in ANSYS, by the numerical simulation.

Tab. 2.1: Natural frequencies for the beam in Example 1 in section 2.2.

j	calculated [rad s^{-1}]	calculated [Hz]	ANSYS [Hz]
1	41.55	6.61	6.60
2	260.37	41.44	41.27
3	729.06	116.03	115.34
4	1428.66	227.38	225.52
5	2361.67	375.87	371.76
6	3527.93	561.49	553.41

Figures 2.4-2.6 present the second, fourth and sixth mode shapes for the beam in 2.2.2, but modelled in ANSYS.

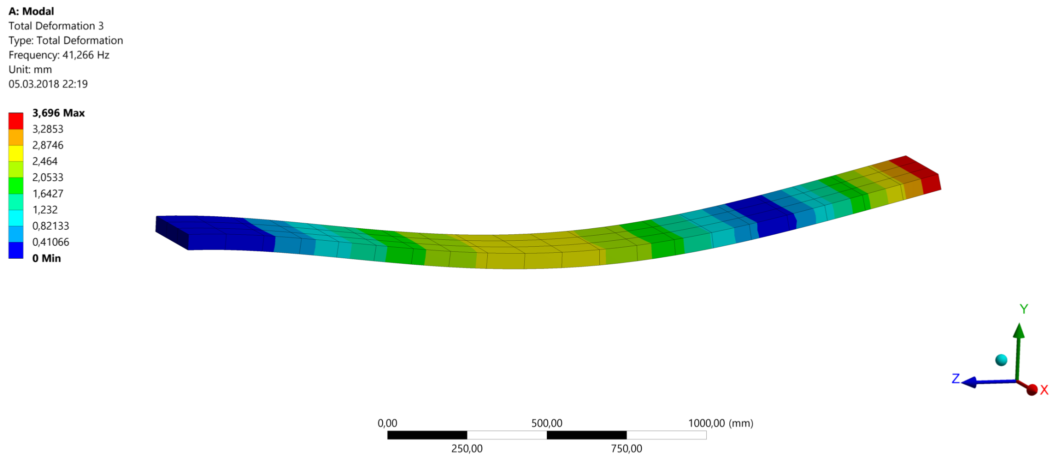


Fig. 2.4: The econd mode shape of the beam in Example 1 in ANSYS.

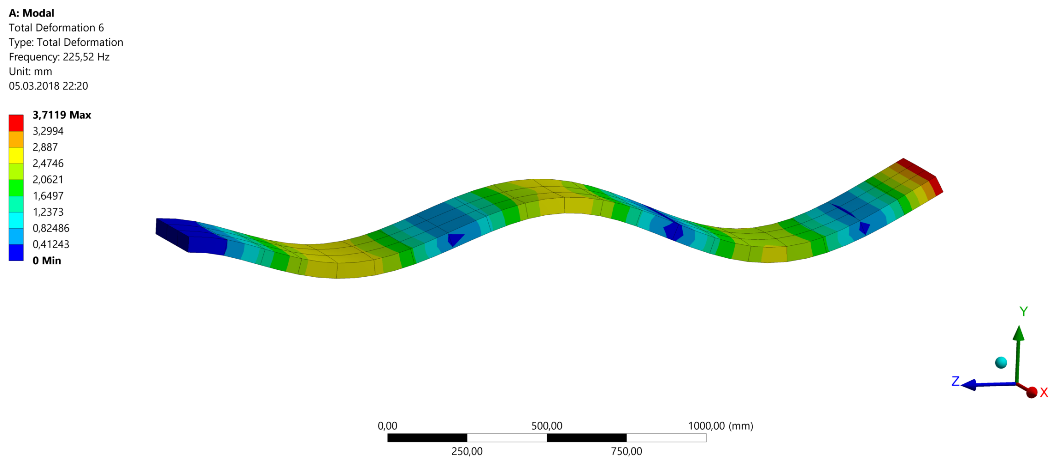


Fig. 2.5: The fourth mode shape of the beam in Example 1 in ANSYS.

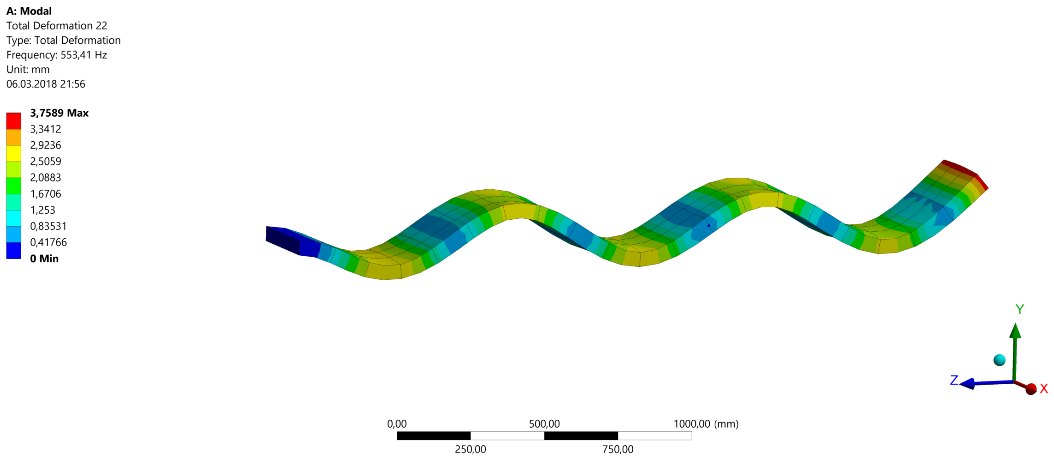


Fig. 2.6: The sixth mode shape of the beam in Example 1 in ANSYS.

We will consider the external damping, which implies $k_e \neq 0$. In this case, the external force will be the Dirac impulse at the time q and the magnitude of $f_0 = 1000$ N; thus the convolution integral in (2.29) can be written as

$$\int_0^t \mathrm{e}^{-\upsilon(t-\tau)} \sin\Big(\Omega_j\,(t-\tau)\Big)\delta(\tau-q)\,\mathrm{d}\tau = \mathrm{e}^{-\upsilon(t-q)} \sin\Big(\Omega_j\,(t-q)\Big)\,\mathrm{H}(t-q). \tag{2.30}$$

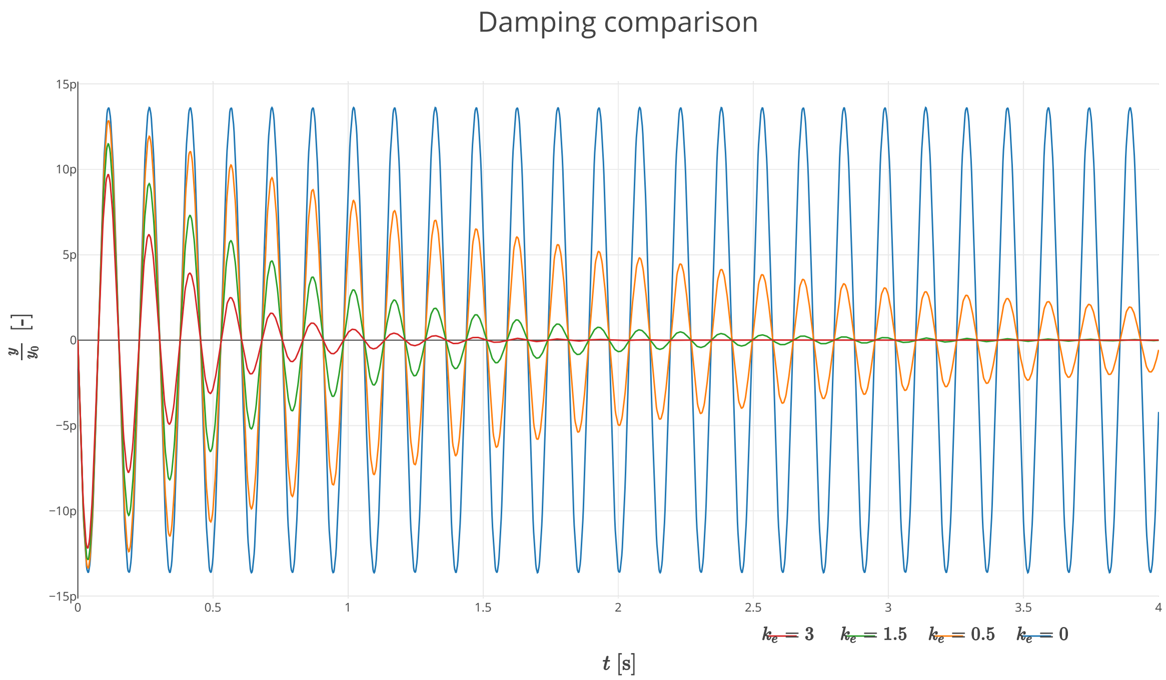


Fig. 2.7: Time paths of the first mode shape for the right end of the undamped and damped beams, with damping coefficients k_e.

The Figure 2.7 shows impulse responses (for $q = 0$ in (2.30)) time paths of oscillations of the right end of the beam for four different damping values $k_e = 0, k_e = 0.5, k_e = 1.5$ and $k_e = 3$. As we can see, the impulse characteristic of the undamped ($k_e = 0$) beam has the shape of a sinusoid with a non-decreasing amplitude. In the case of damped beam, there is a clear trend of decreasing amplitude of right beam deflection due to the energy dissipation. This is caused due to the non-zero real part of the roots λ_1, actually the Re $\lambda_1 < 0$. For example, at time $t = 1.7\,\mathrm{s}$, the deflection amplitude of the beam with $k_e = 0.5$ (orange) is approximtely half, in the case of $k_e = 1.5$ (green) is $\frac{1}{7}$ of the original amplitude and in the last option (red) is almost gone.

2.2.3 Forced harmonic vibration

In this subsection, we consider external loadings at the *RHS* end of the beam with the harmonic time course. Let us take the external harmonic load of the undamped beam in the form $F(t) = f_0 \cos(\Omega\, t)$. The convolution integral from the equation (2.29)

$$\int_0^t \sin\Big(\Omega_j\,(t-\tau)\Big) f_0 \cos(\Omega\,\tau)\mathrm{d}\tau = \frac{\Omega_j\Big(\cos(\Omega\,t) - \cos(\Omega_j\,t)\Big)}{\Omega_j^2 - \Omega^2},$$

thus substituting this convolution back into the equation (2.29) gives

$$y(x,t) = \sum_j \begin{bmatrix} \frac{U(\lambda_j x)}{\lambda_j^2} & \frac{V(\lambda_j x)}{\lambda_j^3} \end{bmatrix} \begin{bmatrix} S(\lambda_j L) & -\frac{T(\lambda_j L)}{\lambda_j} \\ -\lambda_j V(\lambda_j L) & S(\lambda_j L) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot$$
$$\frac{4}{L} \frac{1}{\frac{S(\lambda_j L)V(\lambda_j L) - T(\lambda_j L)U(\lambda_j L)}{\lambda_j^3}} \frac{1}{A\rho} \frac{\Big(\cos(\Omega\,t) - \cos(\Omega_j\,t)\Big)}{\Omega_j^2 - \Omega^2}. \quad (2.31)$$

The Figure 2.8 shows one period of the beam from the Example 2.2, which is loaded with the external load $F(t) = 100\,\cos(2\,t)$. We can notice that the right end of the beam harmoniously moves under the influence of the load, but some small oscillations arise as a result of the so-called beats.

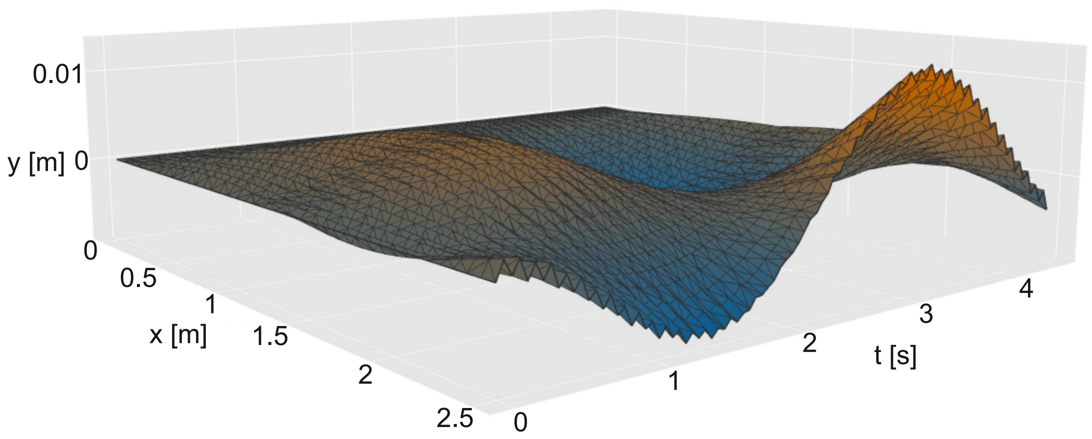


Fig. 2.8: Graph of one oscillation period in 2.2.3. .

As we focus on the last member of the equation (2.31), especially $\Omega_j^2 - \Omega^2$ in the denominator, we suspect that this one can cause some problems, which will depend on the frequency of the external load Ω.

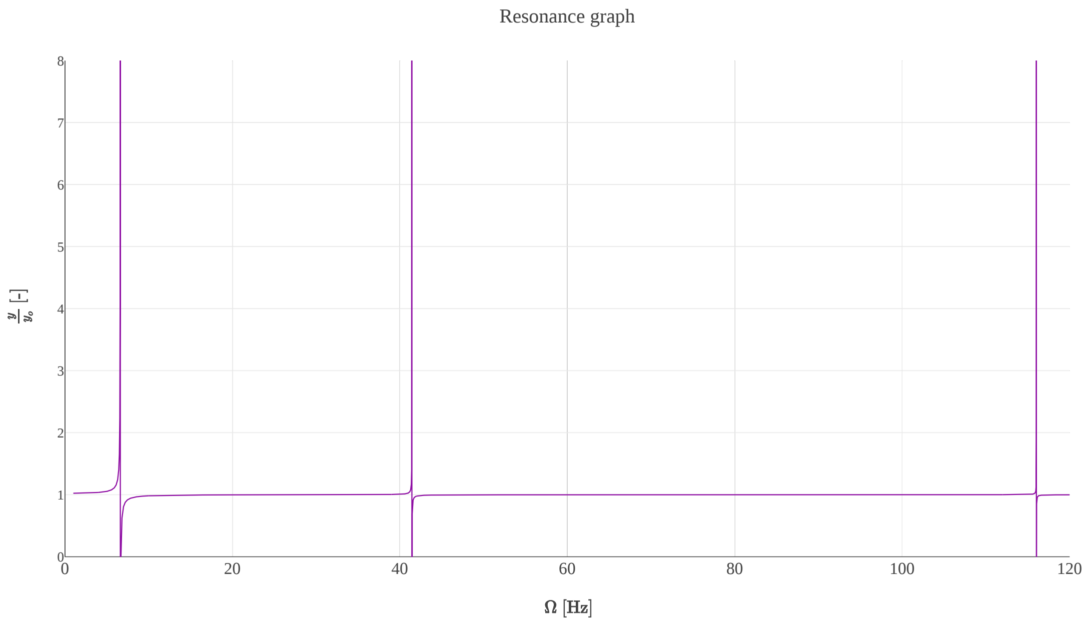


Fig. 2.9: Resonance graph for $\Omega \in [1, 120]$ Hz.

What is possible to see in the Figure 2.9, from the mathematical point of view is that, as the external harmonic frequency Ω is getting closer to the natural frequencies of the beam (in the Table 2.1) the amplitude grows, due to the fact that the denominator of the term

$$\frac{\Big(\cos(\Omega\, t) - \cos(\Omega_j\, t)\Big)}{\Omega_j^2 - \Omega^2},$$

in (2.31) is getting closer to 0, what causes the growth of the amplitude beyond all the limits.

This phenomenon can cause unpleasant damage to real structures. Engineers have this problem in their mind to ensure that the resonance frequencies do not equal the operating frequency to avoid the so-called resonance disaster.

2.3 Second beam example

The second example is similar to the first one; thus, we have a beam on a flexible base with all the parameters same as before. The difference is in the boundary conditions. The right end of the beam is free, but we will model the placement of the left one with pair of springs. The constants k_{o0}, k_{p0} characterise torsional and translational spring respectively. This situation is illustrated in the Figure 2.10.

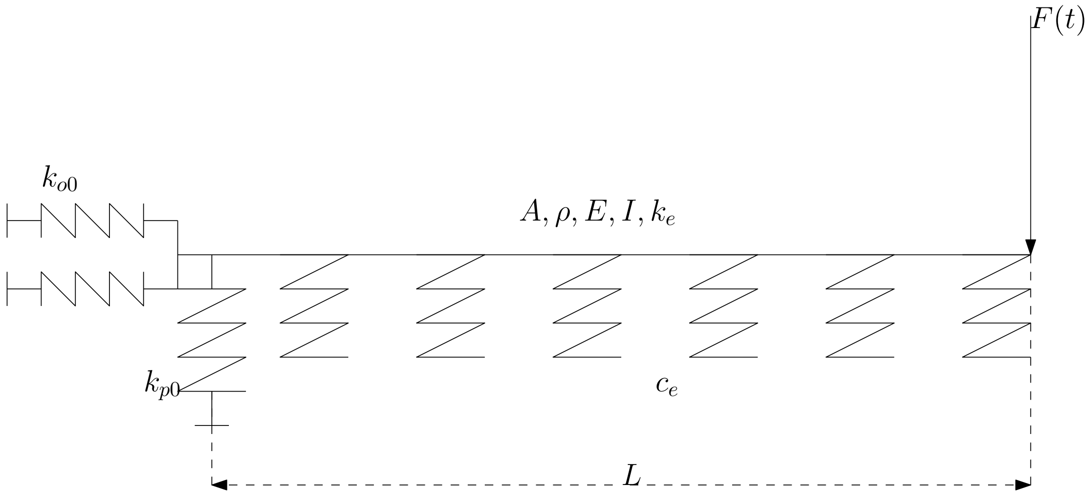


Fig. 2.10: Situation in the beam Example 2 in 2.3.

There are two ways how to formulate and write down the equation and boundary conditions for this case. The first one is more formal, and the other one will be more useful in the next section where we will introduce the general model.

- The equation in the first case

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{F(t)\delta(x-L)}{EI}, \tag{2.32}$$

  with boundary conditions

$$EIy''(0,t) = k_{o0}y'(0) \qquad EIy'''(0,t) = -k_{p0}y(0) \tag{2.33}$$
$$y''(L,t) = 0 \qquad y'''(L,t) = 0, \tag{2.34}$$

  and initial conditions

$$y(x,0) = 0 \qquad \dot{y}(x,0) = 0. \tag{2.35}$$

Applying the *Laplace transform* to the equation (2.32) and using initial conditions (2.35) gives

$$\left(\varepsilon^4 + \frac{\rho A s^2 + k_e s + c_e}{EI}\right)\{y\}_{x,t\to\varepsilon,s} = \varepsilon^3\,\{y(0,t)\}_{t\to s} + \varepsilon^2\,\{y'(0,t)\}_{t\to s} + \varepsilon\,\{y''(0,t)\}_{t\to s} + \{y'''(0,t)\}_{t\to s} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L}. \quad (2.36)$$

Expressing the terms $y''(0,t), y'''(0,t)$ using the boundary conditions (2.33) for the left end of the beam gives for the *RHS* of the previous equation

$$\begin{bmatrix}\varepsilon^3 & \varepsilon^2\end{bmatrix}\mathbf{Y} + \varepsilon\,\frac{k_{o0}}{EI}\,\{y'(0,t)\}_{t\to s} + \frac{-k_{p0}}{EI}\,\{y(0,t)\}_{t\to s} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L}, \quad (2.37)$$

where in this case $\mathbf{Y} = \left\{\begin{bmatrix}y(0,t)\\ y'(0,t)\end{bmatrix}\right\}_{t\to s}$. Thus, the (2.36) can be rewritten as

$$\left(\varepsilon^4 + \frac{\rho A s^2 + k_e s + c_e}{EI}\right)\{y\}_{x,t\to\varepsilon,s} = \begin{bmatrix}\varepsilon^3 - \frac{k_{p0}}{EI} & \varepsilon^2 + \varepsilon\frac{k_{o0}}{EI}\end{bmatrix}\mathbf{Y} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L}. \quad (2.38)$$

- The equation for the second approach is

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{F(t)\delta(x-L)}{EI} - \frac{k_{p0}y(0,t)\delta(x-0)}{EI} + \frac{k_{o0}y'(0,t)\delta'(x-0)}{EI}, \quad (2.39)$$

with boundary conditions for the beam with free ends

$$y''(0,t) = 0 \qquad y'''(0,t) = 0 \quad (2.40)$$
$$y''(L,t) = 0 \qquad y'''(L,t) = 0, \quad (2.41)$$

and initial conditions

$$y(x,0) = 0 \qquad \dot{y}(x,0) = 0. \quad (2.42)$$

Laplace transform applied to the equation (2.39) with the conditions (2.40) and (2.42) leads us to

$$\left(\varepsilon^4 + \frac{\rho A s^2 + k_e s + c_e}{EI}\right)\{y\}_{x,t\to\varepsilon,s} = \begin{bmatrix}\varepsilon^3 & \varepsilon^2\end{bmatrix}\mathbf{Y} + \frac{\{F(t)\}_{t\to s}}{EI}\mathrm{e}^{-\varepsilon L} - \frac{k_{p0}\{y(0,t)\}_{t\to s}\mathrm{e}^{-\varepsilon 0}}{EI} + \frac{k_{o0}\{y'(0,t)\}_{t\to s}\varepsilon\mathrm{e}^{-\varepsilon 0}}{EI}, \quad (2.43)$$

where $\mathbf{Y}$ is again $\left\{\begin{bmatrix}y(0,t)\\ y'(0,t)\end{bmatrix}\right\}_{t\to s}$. It is easy to see, that we can rewrite (2.43) into the same equation as (2.38).

2.3.1 Solution of the second example

In both cases, we have reached the same equation so we will prefer the second option, which gives us more general insight into the problem. As in the first example, we denote

$$\lambda^4 = -\frac{\rho A s^2 + k_e s + c_e}{EI},$$

thus the equation (2.38) turns into

$$\begin{aligned}\{y\}_{x,t\to\varepsilon,s} = \frac{1}{\varepsilon^4-\lambda^4}\left(\begin{bmatrix}\varepsilon^3 - \frac{k_{p0}}{EI} & \varepsilon^2 + \varepsilon\frac{k_{o0}}{EI}\end{bmatrix}\mathbf{Y} + \frac{\{F(t)\}_{t\to s}\mathrm{e}^{-\varepsilon L}}{EI}\right) = \\ \left\{\begin{bmatrix}S(\lambda x) - \frac{k_{p0}}{EI}\frac{V(\lambda x)}{\lambda^3}\mathrm{H}(x) & \frac{T(\lambda x)}{\lambda} + \frac{k_{o0}}{EI}\frac{U(\lambda x)}{\lambda^2}\mathrm{H}(x)\end{bmatrix}\right\}_{x\to\varepsilon}\mathbf{Y} + \\ \left\{\frac{V\Big(\lambda(x-L)\Big)}{\lambda^3}\mathrm{H}(x-L)\right\}_{x\to\varepsilon}\frac{\{F(t)\}_{t\to s}}{EI}.\end{aligned} \tag{2.44}$$

Applying the boundary condition (2.41) for the right end of the beam to the equation (2.44) gives

$$\begin{aligned}\begin{bmatrix}\mathcal{E}_L^2\\ \mathcal{E}_L^3\end{bmatrix}\{y\}_{x,t\to\varepsilon,s} = \begin{bmatrix}0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}\lambda^2 U(\lambda L) - \frac{k_{p0}}{EI}\frac{T(\lambda L)}{\lambda} & \lambda V(\lambda L) + \frac{k_{o0}}{EI}S(\lambda L)\\ \lambda^3 T(\lambda L) - \frac{k_{p0}}{EI}S(\lambda L) & \lambda^2 U(\lambda L) + \lambda\frac{k_{o0}}{EI}V(\lambda L)\end{bmatrix}}_{\diamondsuit}\mathbf{Y} + \\ \begin{bmatrix}\frac{T\Big(\lambda(L-L)\Big)}{\lambda}\\ S\Big(\lambda(L-L)\Big)\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}.\end{aligned} \tag{2.45}$$

Thus, we express $\mathbf{Y} = -\diamondsuit^{-1}\begin{bmatrix}0\\1\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}$ from (2.45) and get

$$\{y\}_{x,t\to\varepsilon,s} = -\left\{\begin{bmatrix}S(\lambda x) - \frac{k_{p0}}{EI}\frac{V(\lambda x)}{\lambda^3} & \frac{T(\lambda x)}{\lambda} + \frac{k_{o0}}{EI}\frac{U(\lambda x)}{\lambda^2}\end{bmatrix}\right\}_{x\to\varepsilon}\diamondsuit^{-1}\begin{bmatrix}0\\1\end{bmatrix}\frac{\{F(t)\}_{t\to s}}{EI}. \tag{2.46}$$

We have reached the same point, with different values, as in (2.23) and so the next procedure is the same and will not be discussed here further. However, we will focus on comparing this example with the example given in the section 2.2.

2.3.2 Natural frequencies

We would like to compare the calculated natural frequencies for the beam from the section 2.2 and the accuracy of spring-modelling of the cantilevered beam in this subchapter. To determine natural beam frequencies, we have to express the determinant of $\diamondsuit$ from the equation (2.45).

$$\det(\diamondsuit) = -0.5\lambda^4 \cos(L\lambda)\cosh(L\lambda) + 0.5\lambda^4 - \frac{0.5k_{p0}}{EI}\lambda \sin(L\lambda)\cosh(L\lambda) + \frac{0.5k_{p0}}{EI}\lambda \cos(L\lambda)\sinh(L\lambda) - \frac{0.5k_{o0}}{EI}\lambda^3 \sin(L\lambda)\cosh(L\lambda) - \frac{0.5k_{o0}}{EI}\lambda^3 \cos(L\lambda)\sinh(L\lambda) + \frac{0.5k_{p0}}{EI^2}k_{o0}\cos(L\lambda)\cosh(L\lambda) + \frac{0.5k_{p0}}{EI^2}k_{o0}. \quad (2.47)$$

The cantilever beam is the limit case, so we have to consider sufficiently large springs coefficients k_{p0}, k_{o0} for its modelling.

Tab. 2.2: Natural frequencies of cantilevered beam and its models. [Hz].

j	Values in Example 1	$k_{p0} = 9.99 \times 10^9$ $k_{o0} = 9.99 \times 10^7$	$k_{p0} = 9.99 \times 10^{10}$ $k_{o0} = 9.99 \times 10^8$
1	6.61	6.53	6.61
2	41.44	40.81	41.43
3	116.03	113.73	116.01
4	227.38	221.11	227.04
5	375.87	361.16	-
6	561.49	537.93	-

Natural frequencies of the different beams

It is easy to see that if we set both spring coefficients $k_{p0} = k_{o0} = 0$, we have a model of the beam with free ends. From the equation (2.47) we can get the characteristic roots and obtain the natural frequencies.

Tab. 2.3: Natural frequencies of the beam with free ends. [Hz]

j	Calculated (E-B)	ANSYS
1	42.08	41.49
2	115.99	114.27
3	227.38	223.75
4	375.87	369.17

To get a better and more precise idea of how the stiffness of springs is related to the natural frequencies, we express this dependency graphically. Let us consider *LHS* cantilevered beam, where the translational spring acts on the right end. The Table 2.4 lists the natural frequencies of such a beam and the Figures 2.11 and 2.12 visualised it.

Tab. 2.4: Cantilever beam with the translational spring on the right end. [Hz]

$k_{pL-\mathrm{rel}}$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	6.61	41.44	116.03	227.38	375.87
0.1	9.71	42.08	116.26	227.49	375.94
0.3	13.58	43.43	116.72	227.73	376.08
0.7	18.04	46.26	117.67	228.2	376.37
1.2	21.12	49.85	118.91	228.8	376.72
8	27.57	76.98	138.46	238.15	381.95
∞	29.01	93.97	196.05	335.26	511.82

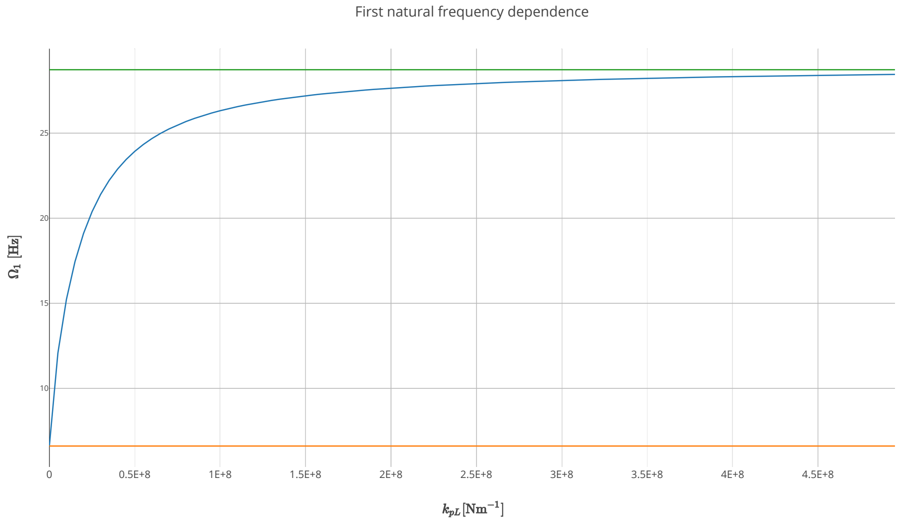


Fig. 2.11: First natural frequency dependence on the stiffness of the translational spring.

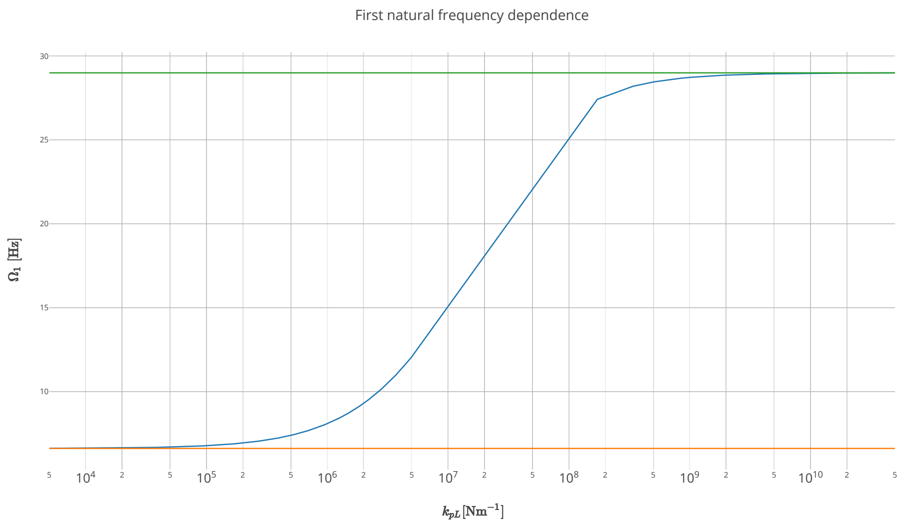


Fig. 2.12: First natural frequency dependence on the stiffness of the spring. [log]

2.4 General beam model

Based on the previous examples, we create a general model for an arbitrary loaded beam. In the model, we will consider the effects of the external loads in n points G_i, for $i = 1, \ldots, n$, on the beam of the length L. The left end of the beam is denoted as G_0 and the right one G_L. In every point G_i can be any of the loads from the table 2.5, where g_i is the location of the point G_i.

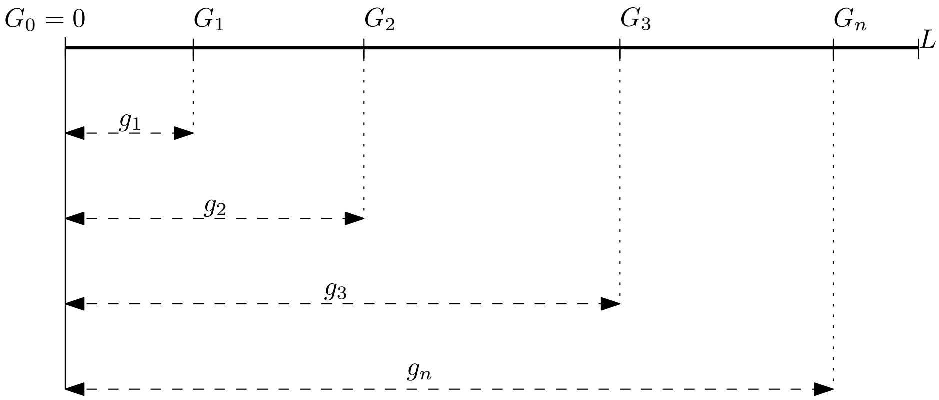


Fig. 2.13: Generally loaded beam with loads from the Table 2.5.

According to the section 2.3.1 we consider a beam with free ends, where various situations will be modeled by relevant spring coefficients k_{pi}, k_{oi} with boundary conditions as

$$y''(0,t) = 0 \qquad y'''(0,t) = 0 \tag{2.48}$$

$$y''(L,t) = 0 \qquad y'''(L,t) = 0. \tag{2.49}$$

It is assumed that the beam has no initial deflection and that angle of rotation is zero at the initial time, thus

$$y(x,0) = 0 \qquad \dot{y}(x,0) = 0. \tag{2.50}$$

We start with the general equation (2.13) derived in the section 2.1

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{q_v}{EI} - \frac{m_v'}{EI}. \tag{2.51}$$

Tab. 2.5: Table of the beam loads.

Type	
Force	$F(t)\,\delta(x-g_i)$
Moment	$M(t)\,\delta(x-g_i)$
Translational spring	$-k_{pi}\,y(g_i,t)\,\delta(x-g_i)$
Torsional spring	$k_{oi}\,y'(g_i,t)\delta(x-g_i)$

From equation (2.51) and the Table 2.5 we get[2]

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \sum_{i=0}^{k}\frac{F_i(t)\,\delta(x-g_i)}{EI} + \sum_{i=0}^{k}\frac{M_i(t)\,\delta'(x-g_i)}{EI} - \sum_{i=0}^{k}\frac{k_{pi}\,y(g_i,t)\,\delta(x-g_i)}{EI} + \sum_{i=0}^{k}\frac{k_{oi}\,y'(g_i,t)\,\delta'(x-g_i)}{EI}. \quad (2.52)$$

Remark. In the equation (2.52) we sum up to $k = \begin{cases} n-1 & \text{if there is load at } G_0 \\ n & \text{if there is no load at } G_0 \end{cases}$

E	*Young's modulus*
I	Area moment of inertia of the beam
$F_i(t)$	The time-dependent function of the force loaded at the point Q_i
$M_i(t)$	The time-dependent function of the moment loaded at the point Q_i
k_{pi}	Stiffness coefficient of the translational spring loaded at the point Q_i
k_{oi}	Stiffness coefficient of the torsional spring loaded at the point Q_i
g_i	Position of the point G_i
ρ	Density per unit length of the beam
A	Cross-sectional area of the beam
k_e	External damping coefficient
c_e	Flexible subsoil coefficient

2.4.1 Derivation of the general beam model

As in the previous examples, we apply the double *Laplace transform*, denote $\lambda^4 = -\frac{\rho A s^2 + k_e s + c_e}{EI}$ and from (2.52) we get

$$\{y(x,t)\}_{x,t\to\varepsilon,s} = \frac{1}{\varepsilon^4-\lambda^4}\left(\left[\varepsilon^3 - \tfrac{k_{p0}}{EI} \quad \varepsilon^2 + \varepsilon\tfrac{k_{o0}}{EI}\right]\mathbf{Y} + \sum_{i=0}^{k}\left(\frac{\{F_i(t)\}_{t\to s}\mathrm{e}^{-\varepsilon g_i}}{EI} + \frac{\varepsilon\{M_i(t)\}_{t\to s}\mathrm{e}^{-\varepsilon g_i}}{EI}\right) + \sum_{i=1}^{k}\left(-\frac{k_{pi}\{y(g_i,t)\}_{t\to s}\mathrm{e}^{-\varepsilon g_i}}{EI} + \frac{\varepsilon k_{oi}\{y'(g_i,t)\}_{t\to s}\mathrm{e}^{-\varepsilon g_i}}{EI}\right)\right) =$$

$$\left\{\left[S(\lambda x) - \tfrac{V(\lambda x)}{\lambda^3}\tfrac{k_{p0}}{EI}\mathrm{H}(x) \quad \tfrac{T(\lambda x)}{\lambda} + \tfrac{U(\lambda x)}{\lambda^2}\tfrac{k_{o0}}{EI}\mathrm{H}(x)\right]\right\}_{x\to\varepsilon}\mathbf{Y} + \sum_{i=0}^{k}\left(\left\{\frac{V\big(\lambda(x-g_i)\big)}{\lambda^3}\mathrm{H}(x-g_i)\right\}_{x\to\varepsilon}\cdot\frac{\{F_i(t)\}_{t\to s}}{EI} + \left\{\frac{U\big(\lambda(x-g_i)\big)}{\lambda^2}\mathrm{H}(x-g_i)\right\}_{x\to\varepsilon}\frac{\{M_i(t)\}_{t\to s}}{EI}\right) + \sum_{i=1}^{k}\left(-\left\{\frac{V\big(\lambda(x-g_i)\big)}{\lambda^3}\mathrm{H}(x-g_i)\right\}_{x\to\varepsilon}\cdot\frac{k_{pi}\{y(g_i,t)\}_{t\to s}}{EI} + \left\{\frac{U\big(\lambda(x-g_i)\big)}{\lambda^2}\mathrm{H}(x-g_i)\right\}_{x\to\varepsilon}\frac{k_{oi}\,\{y'(g_i,t)\}_{t\to s}}{EI}\right), \quad (2.53)$$

where $\mathbf{Y} = \left\{\begin{bmatrix} y(0,t) \\ y'(0,t)\end{bmatrix}\right\}_{t\to s}$. Applying the boundary conditions (2.49) for the right end of the beam gives us last two rows of the matrices $\mathbf{Z}$ and $\mathbf{J}$, which will be introduced later in (2.59) and (2.61) respectively.

[2]Derivatives of the moment-kind loads from the table 2.5, in equation (2.52) arise due to the term $\frac{m_v'}{EI}$.

$$\begin{bmatrix}\mathcal{E}_L^2\\ \mathcal{E}_L^3\end{bmatrix}\{y(x,t)\}_{x,t\to\varepsilon,s}=\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}\lambda^2U(\lambda L)-\frac{T(\lambda L)}{\lambda}\frac{k_{p0}}{EI} & \lambda V(\lambda L)+S(\lambda L)\frac{k_{o0}}{EI}\\ \lambda^3T(\lambda L)-S(\lambda L)\frac{k_{p0}}{EI} & \lambda^2U(\lambda L)+\lambda V(\lambda L)\frac{k_{o0}}{EI}\end{bmatrix}\mathbf{Y}+$$
$$\sum_{i=0}^{k}\left(\begin{bmatrix}\frac{T\left(\lambda(L-g_i)\right)}{\lambda}\\ S\left(\lambda(L-g_i)\right)\end{bmatrix}\frac{\{F_i(t)\}_{t\to s}}{EI}+\begin{bmatrix}S\left(\lambda(L-g_i)\right)\\ \lambda V\left(\lambda(L-g_i)\right)\end{bmatrix}\frac{\{M_i(t)\}_{t\to s}}{EI}\right)+$$
$$\sum_{i=1}^{k}\left(-\begin{bmatrix}\frac{T\left(\lambda(L-g_i)\right)}{\lambda}\\ S\left(\lambda(L-g_i)\right)\end{bmatrix}\frac{k_{pi}\{y(g_i,t)\}_{t\to s}}{EI}+\begin{bmatrix}S\left(\lambda(L-g_i)\right)\\ \lambda V\left(\lambda(L-g_i)\right)\end{bmatrix}\frac{k_{oi}\{y'(g_i,t)\}_{t\to s}}{EI}\right).\quad(2.54)$$

In order to simplify the further expressions, we use the two following general equations, with the help of operator defined in the section 1.3.2. Hence, the equation (2.53) in the point G_h is

$$\mathcal{E}_{g_h}\{y(x,t)\}_{x,t\to\varepsilon,s}=\left[S(\lambda g_h)-\frac{V(\lambda g_h)}{\lambda^3}\frac{k_{p0}}{EI}\quad \frac{T(\lambda g_h)}{\lambda}+\frac{U(\lambda g_h)}{\lambda^2}\frac{k_{o0}}{EI}\right]\mathbf{Y}+$$
$$\sum_{i=0}^{k}\left(\frac{V\left(\lambda(g_h-g_i)\right)}{\lambda^3}\,\mathrm{H}(g_h-g_i)\,\frac{\{F_i(t)\}_{t\to s}}{EI}+\frac{U\left(\lambda(g_h-g_i)\right)}{\lambda^2}\,\mathrm{H}(g_h-g_i)\,\frac{\{M_i(t)\}_{t\to s}}{EI}\right)+$$
$$\sum_{i=1}^{k}\left(-\frac{V\left(\lambda(g_h-g_i)\right)}{\lambda^3}\,\mathrm{H}(g_h-g_i)\,\frac{k_{pi}\{y(g_i,t)\}_{t\to s}}{EI}+\frac{U\left(\lambda(g_h-g_i)\right)}{\lambda^2}\,\mathrm{H}(g_h-g_i)\,\frac{k_{oi}\{y'(g_i,t)\}_{t\to s}}{EI}\right).\quad(2.55)$$

Moreover, the first derivative of this equation at the same point

$$\mathcal{E}_{g_h}^1\{y(x,t)\}_{x,t\to\varepsilon,s}=\left[\lambda V(\lambda g_h)-\frac{U(\lambda g_h)}{\lambda^2}\frac{k_{p0}}{EI}\quad S(\lambda g_h)+\frac{T(\lambda g_h)}{\lambda}\frac{k_{o0}}{EI}\right]\mathbf{Y}+$$
$$\sum_{i=0}^{k}\left(\frac{U\left(\lambda(g_h-g_i)\right)}{\lambda^2}\,\mathrm{H}(g_h-g_i)\,\frac{\{F_i(t)\}_{t\to s}}{EI}+\frac{T\left(\lambda(g_h-g_i)\right)}{\lambda}\,\mathrm{H}(g_h-g_i)\,\frac{\{M_i(t)\}_{t\to s}}{EI}\right)+$$
$$\sum_{i=1}^{k}\left(-\frac{U\left(\lambda(x-g_i)\right)}{\lambda^2}\,\mathrm{H}(g_h-g_i)\,\frac{k_{pi}\{y(g_i,t)\}_{t\to s}}{EI}+\left\{\frac{T\left(\lambda(x-g_i)\right)}{\lambda}\right\}\mathrm{H}(g_h-g_i)\,\frac{k_{oi}\{y'(g_i,t)\}_{t\to s}}{EI}\right).\quad(2.56)$$

2.4.2 General beam model matrices

We focus on the number of unknown coefficients and the number of equations required for expressing them. Due to the nature of the *Laplace transform*, we get two unknown coefficient $y(0,t)$ and $y'(0,t)$ from the equation (2.53). The other unknown deflections $y(g_i,t)$ and rotations $y'(g_i,t)$ coefficients arise at the points G_i with the translational or torsional springs respectively.

Thus, in total, we have $2(k+1)$ unknown coefficients, which we have to express and substitute back into the equation (2.53). We rewrite the system of equations (2.54), (2.55), (2.56) into a matrix equation, with the vector of unknown coefficients denoted as $\widehat{\mathbf{Y}}$.

$$\widehat{\mathbf{Y}} = \left\{ \left\{ \begin{bmatrix} y(0,t) \\ y'(0,t) \\ y(q_1,t) \\ y'(q_1,t) \\ \vdots \\ y(q_n,t) \\ y'(q_n,t) \end{bmatrix} \right\}_{t \to s} \right\} \quad 2(k+1). \tag{2.57}$$

The previously mentioned matrix equations can be written as

$$\mathbf{Z} \cdot \widehat{\mathbf{Y}} = \mathbf{J}, \tag{2.58}$$

where $\mathbf{J}$ is introduced in (2.61) and $\mathbf{Z}$ follows.

The general expression of the matrix $\mathbf{Z}$ is quite complicated, but there exists certain regularity. The first $2k$ rows of the matrix are expressed from the equations (2.55), (2.56) and the last two from (2.54). Matrix $\mathbf{Z}$ is a block matrix such that

$$\mathbf{Z} = \begin{bmatrix} \mathbf{A_1} & -\mathbb{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A_2} & \mathbf{B_{2,1}} & -\mathbb{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A_3} & \mathbf{B_{3,1}} & \mathbf{B_{3,2}} & -\mathbb{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A_4} & \mathbf{B_{4,1}} & \mathbf{B_{4,2}} & \mathbf{B_{4,3}} & -\mathbb{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{A_k} & \mathbf{B_{k,1}} & \mathbf{B_{k,2}} & \mathbf{B_{k,3}} & \mathbf{B_{k,4}} & \dots & -\mathbb{I} \\ \widehat{\mathbf{C}} & \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} & \mathbf{C_4} & \dots & \mathbf{C_k} \\ \widehat{\mathbf{D}} & \mathbf{D_1} & \mathbf{D_2} & \mathbf{D_3} & \mathbf{D_4} & \dots & \mathbf{D_k} \end{bmatrix}, \tag{2.59}$$

where

$$\begin{aligned} \mathbf{A_i} &= \begin{bmatrix} A_i & A_i^* \\ \widehat{A}_i & \widehat{A}_i^* \end{bmatrix} & \mathbf{B_{i,j}} &= \begin{bmatrix} B_{i,j} & B_{i,j}^* \\ \widehat{B}_{i,j} & \widehat{B}_{i,j}^* \end{bmatrix} \\ \widehat{\mathbf{C}} &= \begin{bmatrix} \widehat{C} & \widehat{C}^* \end{bmatrix} & \widehat{\mathbf{D}} &= \begin{bmatrix} \widehat{D} & \widehat{D}^* \end{bmatrix} \\ \mathbf{C_i} &= \begin{bmatrix} C_i & C_i^* \end{bmatrix} & \mathbf{D_i} &= \begin{bmatrix} D_i & D_i^* \end{bmatrix} \\ \mathbb{I} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathbf{0} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus matrix $\mathbf{Z}$ can be rewritten as

$$\mathbf{Z} = \begin{bmatrix} A_1 & A_1^* & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \widehat{A}_1 & \widehat{A}_1^* & 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ A_2 & A_2^* & B_{2,1} & B_{2,1}^* & -1 & 0 & 0 & 0 & \dots \\ \widehat{A}_2 & \widehat{A}_2^* & \widehat{B}_{2,1} & \widehat{B}_{2,1}^* & 0 & -1 & 0 & 0 & \dots \\ A_3 & A_3^* & B_{3,1} & B_{3,1}^* & B_{3,2} & B_{3,2}^* & -1 & 0 & \ddots \\ \widehat{A}_3 & \widehat{A}_3^* & \widehat{B}_{3,1} & \widehat{B}_{3,1}^* & \widehat{B}_{3,2} & \widehat{B}_{3,2}^* & 0 & -1 & \ddots \\ A_4 & A_4^* & B_{4,1} & B_{4,1}^* & B_{4,2} & B_{4,2}^* & B_{4,3} & B_{4,3}^* & \ddots \\ \widehat{A}_4 & \widehat{A}_4^* & \widehat{B}_{4,1} & \widehat{B}_{4,1}^* & \widehat{B}_{4,2} & \widehat{B}_{4,2}^* & \widehat{B}_{4,3} & \widehat{B}_{4,3}^* & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ A_k & A_k^* & B_{k,1} & B_{k,1}^* & B_{k,2} & B_{k,2}^* & B_{k,3} & B_{k,3}^* & \dots \\ \widehat{A}_k & \widehat{A}_k^* & \widehat{B}_{k,1} & \widehat{B}_{k,1}^* & \widehat{B}_{k,2} & \widehat{B}_{k,2}^* & \widehat{B}_{k,3} & \widehat{B}_{k,3}^* & \dots \\ \widehat{C} & \widehat{C}^* & C_1 & C_1^* & C_2 & C_2^* & C_3 & C_3^* & \dots \\ \widehat{D} & \widehat{D}^* & D_1 & D_1^* & D_2 & D_2^* & D_3 & D_3^* & \cdots \end{bmatrix} \tag{2.60}$$

with the individual terms

$$\begin{aligned} A_i &= S(\lambda g_i) - \tfrac{V(\lambda g_i)}{\lambda^3}\tfrac{k_{p0}}{EI} & B_{i,j} &= -\tfrac{V(\lambda(g_i-g_j))}{\lambda^3}\tfrac{k_{pj}}{EI} \\ \widehat{A}_i &= \lambda V(\lambda g_i) - \tfrac{U(\lambda g_i)}{\lambda^2}\tfrac{k_{p0}}{EI} & \widehat{B}_{i,j} &= -\tfrac{U(\lambda(g_i-g_j))}{\lambda^2}\tfrac{k_{pj}}{EI} \\ A_i^* &= \tfrac{T(\lambda g_i)}{\lambda} + \tfrac{U(\lambda g_i)}{\lambda^2}\tfrac{k_{o0}}{EI} & B_{i,j}^* &= \tfrac{U(\lambda(g_i-g_j))}{\lambda^2}\tfrac{k_{oj}}{EI} \\ \widehat{A}_i^* &= S(\lambda g_i) + \tfrac{T(\lambda g_i)}{\lambda}\tfrac{k_{o0}}{EI} & \widehat{B}_{i,j}^* &= \tfrac{T(\lambda(g_i-g_j))}{\lambda}\tfrac{k_{oj}}{EI} \\ \widehat{C} &= \lambda^2 U(\lambda L) - \tfrac{T(\lambda L)}{\lambda}\tfrac{k_{p0}}{EI} & \widehat{C}^* &= \lambda V(\lambda L) + S(\lambda L)\tfrac{k_{o0}}{EI} \\ C_i &= -\tfrac{T(\lambda(L-g_i))}{\lambda}\tfrac{k_{pi}}{EI} & C_i^* &= S(\lambda(L-g_i))\tfrac{k_{oi}}{Ei} \\ \widehat{D} &= \lambda^3 T(\lambda L) - S(\lambda L)\tfrac{k_{p0}}{EI} & \widehat{D}^* &= \lambda^2 U(\lambda L) + \lambda V(\lambda L)\tfrac{k_{o0}}{EI} \\ D_i &= -S(\lambda(L-g_i))\tfrac{k_{pi}}{EI} & D_i^* &= \lambda V(\lambda(L-g_i))\tfrac{k_{oi}}{EI}. \end{aligned}$$

The matrix of the right sides with forces and moments is

$$\mathbf{J} = \begin{bmatrix} \chi_1 & \widehat{\chi_1} & \chi_2 & \widehat{\chi_2} & \dots & \chi_n & \widehat{\chi_n} & \psi & \widehat{\psi} \end{bmatrix}^{\mathrm{T}}, \tag{2.61}$$

where

$$\begin{aligned} \chi_i &= \sum_{j=0}^{k}\left(\frac{V\Big(\lambda(g_i-g_j)\Big)}{\lambda^3}\,\mathrm{H}(g_i-g_j)\,\frac{\{F_j(t)\}_{t\to s}}{EI} + \frac{U\Big(\lambda(g_i-g_j)\Big)}{\lambda^2}\,\mathrm{H}(g_i-g_j)\,\frac{\{M_j(t)\}_{t\to s}}{EI}\right) \\ \widehat{\chi_i} &= \sum_{j=0}^{k}\left(\frac{U\Big(\lambda(g_i-g_j)\Big)}{\lambda^2}\,\mathrm{H}(g_i-g_j)\,\frac{\{F_j(t)\}_{t\to s}}{EI} + \frac{T\Big(\lambda(g_i-g_j)\Big)}{\lambda}\,\mathrm{H}(g_i-g_j)\,\frac{\{M_j(t)\}_{t\to s}}{EI}\right) \\ \psi &= \sum_{j=0}^{k}\left(\frac{T\Big(\lambda(L-g_j)\Big)}{\lambda}\,\frac{\{F_j(t)\}_{t\to s}}{EI} + S\Big(\lambda(L-g_j)\Big)\,\frac{\{M_j(t)\}_{t\to s}}{EI}\right) \\ \widehat{\psi} &= \sum_{j=0}^{k}\left(S\Big(\lambda(L-g_j)\Big)\,\frac{\{F_j(t)\}_{t\to s}}{EI} + \lambda V\Big(\lambda(L-g_j)\Big)\,\frac{\{M_j(t)\}_{t\to s}}{EI}\right). \end{aligned}$$

It is possible to express the vector of unknowns $\widehat{\mathbf{Y}}$ from the equation (2.58) as

$$\widehat{\mathbf{Y}} = \mathrm{inv}(\mathbf{Z}) \cdot \mathbf{J} = \frac{1}{\det(\mathbf{Z})}\,\mathrm{adj}(\mathbf{Z}) \cdot \mathbf{J}.$$

Substituting the vector of coefficients $\widehat{\mathbf{Y}}$ into (2.53) gives the *Laplace image* of the solution. In order to identify the original in t, it is necessary to decompose the meromorphic function of the parameter λ^4 into the partial fractions and follow the Theorem 3.

2.4.3 General beam model example

To demonstrate how does the general model works, we show how the values of natural frequencies in the Table 2.4 were obtained. We have the equation (2.53) as

$$y'''' + \frac{\rho A}{EI}\ddot{y} + \frac{k_e}{EI}\dot{y} + \frac{c_e}{EI}y = \frac{F_L(t)\,\delta(x-L)}{EI} - \frac{k_{p0}\,y(0,t)\,\delta(x)}{EI} + \frac{k_{o0}\,y'(0,t)\,\delta'(x)}{EI} - \frac{k_{pL}\,y(L,t)\,\delta(L)}{EI} + \frac{k_{oL}\,y'(L,t)\,\delta'(L)}{EI}, \quad (2.62)$$

thus, in this case, we have four unknown coefficients $y(0,t), y'(0,t), y(L,t), y'(L,t)$, which form the matrix $\widehat{\mathbf{Y}}$

$$\widehat{\mathbf{Y}} = \left\{ \begin{bmatrix} y(0,t) \\ y'(0,t) \\ y(L,t) \\ y'(L,t) \end{bmatrix} \right\}_{t \to s}$$

According to the matrix in (2.60) we obtain the matrix $\mathbf{Z}$ as

$$\mathbf{Z} = \begin{bmatrix} S(\lambda L) - \frac{V(\lambda L)}{\lambda^3}\frac{k_{p0}}{EI} & \frac{T(\lambda L)}{\lambda} + \frac{U(\lambda L)}{\lambda^2}\frac{k_{o0}}{EI} & -1 & 0 \\ \lambda V(\lambda L) - \frac{U(\lambda L)}{\lambda^2}\frac{k_{p0}}{EI} & S(\lambda L) + \frac{T(\lambda L)}{\lambda}\frac{k_{o0}}{EI} & 0 & -1 \\ \lambda^2 U(\lambda L) - \frac{T(\lambda L)}{\lambda}\frac{k_{p0}}{EI} & \lambda V(\lambda L) + S(\lambda L)\frac{k_{o0}}{EI} & -\frac{T(\lambda(L-L))}{\lambda}\frac{k_{pL}}{EI} & S(\lambda(L-L))\frac{k_{oL}}{EI} \\ \lambda^3 T(\lambda L) - S(\lambda L)\frac{k_{p0}}{EI} & \lambda^2 U(\lambda L) + \lambda V(\lambda L)\frac{k_{o0}}{EI} & -S(\lambda(L-L))\frac{k_{pL}}{EI} & \lambda V(\lambda(L-L))\frac{k_{oL}}{EI} \end{bmatrix} \quad (2.63)$$

Due to the fact that

$$T(0) = U(0) = V(0) = 0 \qquad\qquad S(0) = 1$$

the matrix $\mathbf{Z}$ in the equation (2.63) is simplified into

$$\mathbf{Z} = \begin{bmatrix} S(\lambda L) - \frac{V(\lambda L)}{\lambda^3}\frac{k_{p0}}{EI} & \frac{T(\lambda L)}{\lambda} + \frac{U(\lambda L)}{\lambda^2}\frac{k_{o0}}{EI} & -1 & 0 \\ \lambda V(\lambda L) - \frac{U(\lambda L)}{\lambda^2}\frac{k_{p0}}{EI} & S(\lambda L) + \frac{T(\lambda L)}{\lambda}\frac{k_{o0}}{EI} & 0 & -1 \\ \lambda^2 U(\lambda L) - \frac{T(\lambda L)}{\lambda}\frac{k_{p0}}{EI} & \lambda V(\lambda L) + S(\lambda L)\frac{k_{o0}}{EI} & 0 & \frac{k_{oL}}{EI} \\ \lambda^3 T(\lambda L) - S(\lambda L)\frac{k_{p0}}{EI} & \lambda^2 U(\lambda L) + \lambda V(\lambda L)\frac{k_{o0}}{EI} & -\frac{k_{pL}}{EI} & 0 \end{bmatrix}. \quad (2.64)$$

The natural frequencies in the Table 2.4 are for the cantilever beam loaded with the translational spring at the *RHS* of the beam, thus we set $k_{oL} = 0$ in the matrix $\mathbf{Z}$ in (2.64). To obtain the natural frequencies we have to find the roots of the characteristic equation, which is the determinant of $\mathbf{Z}$, thus

$$\det(\mathbf{Z}) = -0.5\lambda^4 \cos(L\lambda)\cosh(L\lambda) + 0.5\lambda^4 - \frac{0.5k_{pL}}{EI}\lambda\sin(L\lambda)\cosh(L\lambda) + \frac{0.5k_{pL}}{EI}\lambda\cdot$$
$$\cos(L\lambda)\sinh(L\lambda) - \frac{0.5k_{p0}}{EI}\lambda\sin(L\lambda)\cosh(L\lambda) + \frac{0.5k_{p0}}{EI}\lambda\cos(L\lambda)\sinh(L\lambda) - \frac{0.5k_{o0}}{EI}\lambda^3\cdot$$
$$\sin(L\lambda)\cosh(L\lambda) - \frac{0.5k_{o0}}{EI}\lambda^3\cos(L\lambda)\sinh(L\lambda) + \frac{k_{pL}k_{p0}}{(EI)^2\lambda^2}\sin(L\lambda)\sinh(L\lambda) + \frac{k_{pL}}{(EI)^2}\cdot$$
$$k_{o0}\cos(L\lambda)\cosh(L\lambda) + \frac{0.5k_{p0}}{(EI)^2}k_{o0}\cos(L\lambda)\cosh(L\lambda) + \frac{0.5k_{p0}}{(EI)^2}k_{o0} +$$
$$\frac{0.5k_{pL}k_{p0}k_{o0}}{(EI)^3\lambda^3}\sin(L\lambda)\cosh(L\lambda) - \frac{0.5k_{pL}k_{p0}k_{o0}}{(EI)^3\lambda^3}\cos(L\lambda)\sinh(L\lambda) \quad (2.65)$$

The values of the natural frequencies obtained from the roots of the equation (2.65) are listed in the Table 2.6 together with the roots.

Tab. 2.6: Table of the natural frequencies and roots of the characteristic equation of the cantilever beam with free or supported *RHS* end of the beam, modelled by the springs.

Spring	**coefficients**	**i**	$L\,(\mathrm{Im}\,\lambda_\mathbf{i})$	Ω_i[Hz]
		1	1.874907	6.61
k_{p0}	9.9×10^{11}	2	4.693497	41.43
k_{o0}	9.9×10^{9}	3	7.853449	115.99
k_{pL}	0	4	10.993985	227.31
		5	14.134452	375.72
		1	3.926293	28.99
k_{p0}	9.9×10^{11}	2	7.067790	93.94
k_{o0}	9.9×10^{9}	3	10.208499	195.99
k_{pL}	9.9×10^{11}	4	13.349311	335.15
		5	16.489282	511.35

Following the (2.61) we are able to write down the matrix of the *RHS* $\mathbf{J}$ as

$$\mathbf{J} = \begin{bmatrix} \frac{V(\lambda(L-L))}{\lambda^3}\frac{\{F(t)\}_{t\to s}}{EI} \\ \frac{U(\lambda(L-L))}{\lambda^2}\frac{\{F(t)\}_{t\to s}}{EI} \\ \frac{T(\lambda(L-L))}{\lambda}\frac{\{F(t)\}_{t\to s}}{EI} \\ S(\lambda(L-L))\frac{\{F(t)\}_{t\to s}}{EI} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\{F(t)\}_{t\to s}}{EI} \end{bmatrix}.$$

3 HYDRODYNAMICS

In this chapter, we discuss the possibilities of using generalized functions in hydrodynamics and compare this approach with the standard procedures outlined in the book [9]. At first, we have to derive the basic equations for the pipe, which describe the situation in the piping systems.

3.1 Derivation of the pipe equations

3.1.1 The continuity equation for fluid in the tube

From the law of conservation of mass for the fluid in the volume ΔV, according to the Figure 3.1, we are able to write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta V(t)} \rho(t) \,\mathrm{d}V = 0. \tag{3.1}$$

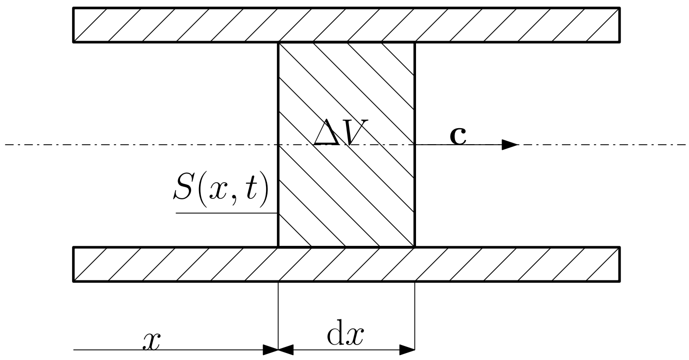


Fig. 3.1: Volume in pipe. Redrawed from [9].

Using so-called *Mean value theorem for definite integrals*[1] gives

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(\gamma)\Delta V] = 0, \text{for specific } \gamma \in \Delta V, \tag{3.2}$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(\gamma)]\Delta V + \rho(\gamma)\frac{\mathrm{d}\Delta V}{\mathrm{d}t} = 0. \tag{3.3}$$

Applying the *Gauss-Ostrogradsky divergence theorem* to the equation (3.3)[2]

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(\gamma)]\Delta V + \rho(\gamma) \int_{S(t)} \mathbf{c}\,\mathbf{n} \,\mathrm{d}S = 0. \tag{3.4}$$

[1]Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $\gamma \in (a, b)$ such that

$$\int_a^b f(x) \,\mathrm{d}x = f(\gamma)(b - a).$$

The same can be applied to higher dimensions.

[2]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta V(t)} \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\Delta V(t) = \int_{\Delta V(t)} \operatorname{div} \mathbf{c} \,\mathrm{d}V$$

The surface is composed of three thought parts, the left S_1 and right S_2 basis, and the side P, as it is shown in the Figure 3.2. Taking into account the orientation of the normal vectors, we can decompose the second term of the equation (3.4) as

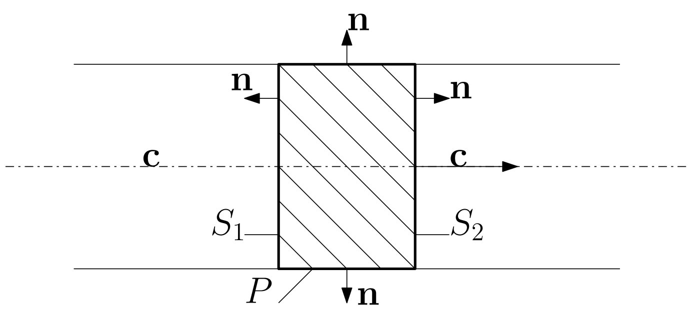


Fig. 3.2: The surface of the considered element. Redrawed from [9].

$$\int_{S(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S = \int_{S_2(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S + \int_{S_1(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S + \int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S = Q_2 - Q_1 + \int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S. \quad (3.5)$$

According to the Figure 3.1 we have $\Delta V(t) = S(x,t)\,\mathrm{d}x$. Similarly to the Chapter 2, for the small $\mathrm{d}x$, we can write $Q_2 = Q_1 + \frac{\partial Q_1}{\partial x}\mathrm{d}x$ and denoting $S_1(t) = S(t), Q_1(t) = Q(t)$ gives for the equation (3.4)

$$\frac{\mathrm{d}\rho(\gamma)}{\mathrm{d}t} S(t) + \rho(\gamma)\frac{\partial Q}{\partial x} + \rho(\gamma)\frac{\int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S}{\mathrm{d}x} = 0. \quad (3.6)$$

In this case, it is useful to rewrite ρ by the relation of the local balance between pressure and density

$$\frac{\mathrm{d}p}{\mathrm{d}t} = a^2\,\frac{\mathrm{d}\rho}{\mathrm{d}t},$$

substituting this identity into (3.6) implies

$$\frac{1}{a^2}\frac{\mathrm{d}p(\gamma)}{\mathrm{d}t} S(t) + \rho(\gamma)\frac{\partial Q}{\partial x} + \rho(\gamma)\frac{\int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S}{\mathrm{d}x} = 0. \quad (3.7)$$

Note. The last member of the equation (3.7) is related to the deformation of the tube and this term have to be determined from the analysis of the deformation of the flexible tube. More details and the entire derivation of the relation can be found in the book [9].

$$\frac{\mathrm{d}p}{\mathrm{d}t}\int_{V_T} S_{ij}\,w_{ij}\,\mathrm{d}V = \int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S. \quad (3.8)$$

As we consider only the thin-walled (w_t is the wall thickness) one-dimensional pipe with the circular cross-section, the internal pressure will cause a unilateral strain in the tube, and we have

$$\Sigma_{ij}v_{ij} = \Sigma_{11}v_{11} = \sigma\frac{\mathrm{d}\epsilon}{\mathrm{d}t}, \quad (3.9)$$

where σ is the tensile stress and ϵ is the fractional extension. Considering all the above mentioned assumptions we get

$$\Sigma_{11} = \sigma = S_{11}p \text{ and } v_{11} = \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = w_{11}\frac{\mathrm{d}p}{\mathrm{d}t}.$$

The normal tension in the tube can be determined[3] as $\sigma = \frac{R}{w_t}p$. The relation between tension and the fractional extension in one dimensional strain is defined by the *Hook's law*, where $\sigma = \epsilon E$, thus we are able to write down a formula for ϵ as

$$\epsilon = \frac{\sigma}{E} = \frac{R}{Ew_t}p \Rightarrow \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{R}{Ew_t}\frac{\mathrm{d}p}{\mathrm{d}t},$$

where we assume, that the tube radius R does not change with time and so we get

$$S_{11}w_{11} = \frac{R^2}{Ew_t^2}. \tag{3.10}$$

We substitute this expression into the equation (3.8)

$$\begin{aligned}\int_{P(t)} \mathbf{c}\,\mathbf{n}\,\mathrm{d}S = \frac{\mathrm{d}p}{\mathrm{d}t}\int_{V_T} \frac{R^2}{Ew_t^2}\,\mathrm{d}V = \frac{\mathrm{d}p}{\mathrm{d}t}\frac{R^2}{Ew_t^2}\int_{V_T}\mathrm{d}V = \\ \frac{\mathrm{d}p}{\mathrm{d}t}\frac{R^2}{Ew_t^2}\mathrm{d}x\Big((\pi(R+w_t)^2 - \pi R^2\Big) = \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}xSD\frac{1}{Ew_t},\end{aligned} \tag{3.11}$$

where the term w_t^2 in the numerator was omitted. Substituting the final expression from the equation (3.11) back into the equation (3.7) gives

$$S\frac{\mathrm{d}p}{\mathrm{d}t}\left[\tfrac{1}{a^2} + \tfrac{\rho(\gamma)D}{Ew_t}\right] + \rho(\gamma)\frac{\partial Q}{\partial x} = 0 \tag{3.12}$$

Denoting $\frac{1}{v^2} = \frac{1}{a^2} + \frac{\rho(\gamma)D}{Ew_t}$ and $E_k = \rho(\gamma)a^2$, where E_k is so-called *fluid elastic modulus*, we can write

$$\frac{1}{K} = \frac{1}{\rho(\alpha)v^2} = \frac{1}{E_k} + \frac{D}{Ew_t}. \tag{3.13}$$

Note. In continuum mechanics, it is necessary to compute the time rate of change of any quantity such as heat, momentum or velocity (which gives acceleration) for a portion of a material moving with a velocity. When the material is fluid in fluid dynamics, the velocity field is the flow velocity, and the quantity of interest might be the pressure of the fluid, which we show as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p(t,x_1,x_2,x_3) &= \frac{\partial p}{\partial t} + \left(\frac{\partial p}{\partial x_1}\right)\left(\frac{\partial x_1}{\partial t}\right) + \left(\frac{\partial p}{\partial x_2}\right)\left(\frac{\partial x_2}{\partial t}\right) + \left(\frac{\partial p}{\partial x_3}\right)\left(\frac{\partial x_3}{\partial t}\right) \\ &= \frac{\partial p}{\partial t} + \left(\frac{\partial p}{\partial x_1}\right)v_{x_1} + \left(\frac{\partial p}{\partial x_2}\right)v_{x_2} + \left(\frac{\partial p}{\partial x_3}\right)v_{x_3} \\ &= \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p \\ &= \frac{\mathrm{D}p}{\mathrm{D}t}.\end{aligned}$$

According to [1], the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\mathrm{grad} \tag{3.14}$$

is called *Material derivative*, *total derivative* or *substantial derivative*.

[3]According to `https://en.wikipedia.org/wiki/Cylinder_stress` .

Substituting (3.13) into (3.12), applying the *material derivative* and rearranging the terms leads to

$$\frac{\partial Q}{\partial x} + \frac{S}{K}\frac{\partial p}{\partial t} + \frac{1}{K}\frac{\partial p}{\partial x}Q = 0, \tag{3.15}$$

which is known as the *continuity equation*.

3.1.2 The equilibrium fluid equation in the tube

The equilibrium equation for the macroscopic particle (equation (12.3) in [9]) is

$$\rho\frac{\mathrm{d}c_i}{\mathrm{d}t} - \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = \rho g_i, \tag{3.16}$$

which we integrate over the volume ΔV. *Gauss-Ostrogradsky divergence theorem* reduces the *volume integrals* to the *surface integrals*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta V(t)} \rho c_i \,\mathrm{d}V - \int_S \pi_{ij} n_j \,\mathrm{d}S + \int_S p n_i \,\mathrm{d}S = \rho g_i \Delta V. \tag{3.17}$$

Choosing the axis x_2 to be parallel to the flow gives for the (3.17)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta V(t)} \rho c_2 \,\mathrm{d}V - \int_S \pi_{2j} n_j \,\mathrm{d}S + \int_S p n_2 \,\mathrm{d}S = \rho g_2 \Delta V. \tag{3.18}$$

Applying the *Mean value theorem for definite integrals* to the first term in (3.18) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta V(t)} \rho c_2 \,\mathrm{d}V = \frac{\mathrm{d}c_2(\gamma)}{\mathrm{d}t}\rho(\gamma)\Delta V(t) + c_2(\gamma)\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho(\gamma)\Delta V(t)\Big], \text{ for specific } \gamma \in \Delta V(t). \tag{3.19}$$

The difference in density is small in volume ΔV, so we write $\rho(\gamma) = \rho(a)$. With the help of the conservation of mass law (3.1), we get the relation for the rate change of the momentum, as we discard the second term in the previous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta V(t)} \rho c_2 \,\mathrm{d}V = \rho(a)\frac{\mathrm{d}c_2(a)}{\mathrm{d}t}\Delta V(t). \tag{3.20}$$

Negligible differences of the pressure in S_1, S_2 implies

$$\int_S p n_2 \,\mathrm{d}S = p_2 S_2 - p_1 S_1, \tag{3.21}$$

where $p_2 = p_1 + \frac{\partial p_1}{\partial x}\,\mathrm{d}x$ and $S_2 = S_1 + \frac{\partial S_1}{\partial x}\,\mathrm{d}x$. We substitute these expressions into (3.21), denote $p_1 = P, S_1 = S$, omit the term including $(\mathrm{d}x)^2$ and get the expression for the third term in (3.18) as

$$\int_S p n_2 \,\mathrm{d}S = \frac{\partial}{\partial x}(pS)\,\mathrm{d}x. \tag{3.22}$$

The second term of (3.18), $\int_S \pi_{2j} n_j \,\mathrm{d}S$, represents the resultant of internal surface forces, depending on the shear and second viscosity of the fluid. It depends on whether the flow is *laminar* or *turbulent* respectively

$$\int_S \pi_{2j} n_j \,\mathrm{d}S = -Sbc_2(a)\,\mathrm{d}x, \text{ where } b = \frac{32\rho\nu}{D^2}, \tag{3.23}$$

$$\int_S \pi_{2j} n_j \,\mathrm{d}S = -Sbc_2(a)\,\mathrm{d}x, \text{ where } b = \frac{\rho}{2D}\lambda c_2(a). \tag{3.24}$$

Expressions (3.23) and (3.24) do not characterize the physical phenomena in the fluid entirely accurately, as they are developed and analogous to the stationary flow.

Note. For low-viscosity substances, such as water, this term does not have an essential role regarding either damping or frequency oscillation.

Substitute (3.20), (3.22) and (3.23) or (3.24) into (3.18), we get

$$\rho(a)S(t)\frac{\mathrm{d}c_2(a)}{\mathrm{d}t} + S(t)bc_2(a) + \frac{\partial}{\partial x}\Big[pS(t)\Big] = \rho g_2 S(t). \tag{3.25}$$

Due to the negligible changes of the cross-sectional area with the time, we are able to write $S(t) = S_0$, where S_0 is the initial area of the cross-section at position x. Since the differences of density ρ are negligible through the S, we have $\rho(a) = \rho(\alpha) = \rho_0$, and as we denote $c_2(a) = c$ we can rewrite (3.25) into

$$\rho_0 S_0 \frac{\mathrm{d}c}{\mathrm{d}t} + bS_0 c + \frac{\partial}{\partial x}(pS_0) = \rho S_0 g_2 \tag{3.26}$$

If the pipe has the constant cross-sectional area $S_0(x) = S_0$, we use the well-known relation $Q = S_0 c$ for the *volume flow rate* and get

$$\rho_0 \frac{\mathrm{d}Q}{\mathrm{d}t} + bQ + S_0 \frac{\partial p}{\partial x} = \rho S_0 g_2, \tag{3.27}$$

which is considered as the *equilibrium fluid flow equation for the elastic tube with constant cross-section* .

3.1.3 Non-stationary flow equations

Before we continue, we spread the flow to stationary, time-independent and non-stationary parts

$$p = p_0(x) + \sigma(x,t) = p_0 + \sigma, \tag{3.28}$$

$$Q = Q_0(x) + q(x,t) = Q_0 + q. \tag{3.29}$$

Substituting (3.28) and (3.29) into (3.15) gives the *continuity equation* as

$$\frac{\partial}{\partial x}(Q_0 + q) + \frac{S_0}{K}\frac{\partial \sigma}{\partial t} + \frac{1}{K}\frac{\partial(p_0 + \sigma)}{\partial x}(Q_0 + q) = 0, \tag{3.30}$$

which can be divided into two parts-*stationary*

$$\frac{\partial Q_0}{\partial x} + \frac{Q_0}{K}\frac{\partial p_0}{\partial x} = 0 \tag{3.31}$$

and *non-stationary*

$$\frac{\partial q}{\partial x} + \frac{S_0}{K}\frac{\partial \sigma}{\partial t} + \frac{1}{K}\left[\frac{\partial p_0}{\partial x}q + \frac{\partial \sigma}{\partial x}Q_0 + \frac{\partial \sigma}{\partial x}q\right] = 0. \tag{3.32}$$

As we do the same procedure with the equation (3.27) we obtain

$$\rho_0 \frac{\mathrm{d}}{\mathrm{d}t}(Q_0 + q) + b(Q_0 + q) + S_0 \frac{\partial(p + \sigma)}{\partial x} = \rho_0 S_0 g_2. \tag{3.33}$$

Applying the operator of *material derivative* (3.14) to the equation (3.33) gives

$$\rho_0\frac{\partial q}{\partial t}+\rho_0\left(\frac{\partial Q_0}{\partial x}+\frac{\partial q}{\partial x}\right)\frac{q}{S_0}+\rho_0\frac{Q_0}{S_0}\frac{\partial q}{\partial x}+bq+S_0\frac{\partial \sigma}{\partial x}=\rho_0 S_0 g_2-\rho_0\frac{Q_0}{S_0}\frac{\partial Q_0}{\partial x}-bQ_0-S_0\frac{\partial p_0}{\partial x}. \tag{3.34}$$

This equation can be as well divided into *stationary part*

$$\rho_0 S_0 g_2-bQ_0-\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\rho_0}{S_0}Q_0^2+S_0 p_0\right)=0 \tag{3.35}$$

and *non-stationary* one

$$\rho_0\frac{\partial q}{\partial t}+\rho_0\left(\frac{\partial Q_0}{\partial x}+\frac{\partial q}{\partial x}\right)\frac{q}{S_0}+\rho_0\frac{Q_0}{S_0}\frac{\partial q}{\partial x}+bq+S_0\frac{\partial \sigma}{\partial x}=0. \tag{3.36}$$

It is known that for an incompressible liquid $K\to+\infty$, so for equation (3.31) we write

$$\lim_{K\to+\infty}\left(\frac{\partial Q_0}{\partial x}+\frac{Q_0}{K}\frac{\partial p_0}{\partial x}\right)=\frac{\partial Q_0}{\partial x}=0\Rightarrow Q_0=\mathit{constant}, \tag{3.37}$$

we use this fact in the equation (3.35) and obtain

$$S_0 p_0+\frac{1}{2}\frac{\rho_0}{S_0}Q_0^2=(\rho_0 S_0 g_2-bQ_0)x+\mathcal{C}, \tag{3.38}$$

where $\mathcal{C}$ is unknown constant of integration.
The equation (3.38) shows that for stationary flow in pipe with constant cross-sectional area, the mean value of pressure is linear function of spatial variable x, thus

$$\frac{\partial p_0}{\partial x}=\rho_0 g_2-b\frac{Q_0}{S_0}. \tag{3.39}$$

Examples of those problems can be found in [10]. However, because this thesis deals and focus on the problems with non-stationary flow, we substitute obtained intermediate results into equations for non-stationary flow (3.32) and (3.36) and obtain following equations.

Non-stationary flow

Continuity equation

$$\frac{\partial q}{\partial x}+\frac{S_0}{K}\frac{\partial \sigma}{\partial t}+\frac{1}{K}\left[\left(\rho_0 g_2-b\frac{Q_0}{S_0}\right)q+\frac{\partial \sigma}{\partial x}Q_0+\frac{\partial \sigma}{\partial x}q\right]=0 \tag{3.40}$$

Equilibrium fluid flow equation

$$\rho_0\frac{\partial q}{\partial t}+\rho_0\frac{\partial q}{\partial x}\frac{q}{S_0}+\rho_0\frac{Q_0}{S_0}\frac{\partial q}{\partial x}+bq+S_0\frac{\partial \sigma}{\partial x}=0 \tag{3.41}$$

As we can see, the equations (3.40) and (3.41) are non-linear, due to the terms $\frac{q}{K}\frac{\partial \sigma}{\partial x}$ and $\rho_0\frac{q}{S_0}\frac{\partial q}{\partial x}$, this fact complicates the solution of these equations. So the next goal is to estimate the sizes of these terms. As we know from the wave equation, the relation with which the stir is spreading depends at the position x on the speed of the spread c_0. With those assumptions, we can write the new dependencies for the variables as

$$\sigma=\sigma\left(t-\frac{x}{c_0}\right)\text{ and }q=q\left(t-\frac{x}{c_0}\right). \tag{3.42}$$

Since we have differentials of the mentioned variables in the equations (3.40) and (3.41), we denote $z = t - \frac{x}{c_0}$, thus we have

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma}{\partial z} \qquad ; \qquad \frac{\partial \sigma}{\partial x} = \left(-\frac{1}{c_0}\right)\frac{\partial \sigma}{\partial z} \Rightarrow \tag{3.43}$$

$$\frac{\partial \sigma}{\partial t} = -c_0 \frac{\partial \sigma}{\partial x} \quad \wedge \quad \frac{\partial q}{\partial t} = -c_0 \frac{\partial q}{\partial x} \tag{3.44}$$

In real low-damped systems we can identify $c_0 = a$ and the changes in time of these quantities are usually bigger than the local changes. Assuming the mentioned facts we estimate the value of

$$w = \frac{S_0}{K}\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x}\frac{q}{K} = \frac{S_0}{K}\frac{\partial \sigma}{\partial t}\left(1 + \frac{\frac{\partial \sigma}{\partial x}}{\frac{\partial \sigma}{\partial t}}\frac{q}{S_0}\right). \tag{3.45}$$

Substituting from (3.44) into (3.45) gives

$$w = \frac{S_0}{K}\frac{\partial \sigma}{\partial t}\left(1 - \frac{c}{a}\right), \tag{3.46}$$

where we denote $M = \frac{c}{a}$, known as the *Mach number*[4]. It is easy to see, that small value of *Mach number*, $M \ll 1$, allows us to omit the non-linear terms in the equations (3.40) and (3.41) so we obtain
Continuity equation

$$\frac{\partial \sigma}{\partial t} + \frac{K}{S_0}\frac{\partial q}{\partial x} + \frac{Q_0}{S_0}\frac{\partial \sigma}{\partial x} + \frac{\rho_0 g_2 - b\frac{Q_0}{S_0}}{S_0} q = 0$$

Equilibrium fluid flow equation

$$\frac{\partial q}{\partial t} + \frac{S_0}{\rho_0}\frac{\partial \sigma}{\partial x} + \frac{Q_0}{S_0}\frac{\partial q}{\partial x} + \frac{b}{\rho_0} q = 0$$

Let us denote $p = \sigma, Q = q$, $\rho = \rho_0, S = S_0$. In order to solve these equations in the next sections, we will, without the loss of generality, omit the terms which affect the stationary flow, thus we finally write down the equations as
Continuity equation

$$\frac{S}{\rho a^2}\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{3.47}$$

Equilibrium fluid flow equation

$$\frac{\rho}{S}\frac{\partial Q}{\partial t} + \frac{\partial p}{\partial x} + \frac{b}{S} Q = 0 \tag{3.48}$$

[4] $\left(M = \frac{\text{local flow velocity with respect to the boundaries}}{\text{speed of sound in the medium}}\right)$

3.2 Standard solution approach

At first, we characterise the flow problem for a simple pipe, and in the following example, we present the procedure of solving a homogeneous system of equations (3.47) and (3.48) for this problem. This procedure will be further extended to the piping systems.

3.2.1 Characteristics of the flow problem in the simple pipe

We have a simple pipe of length L as is shown in Figure 3.3.

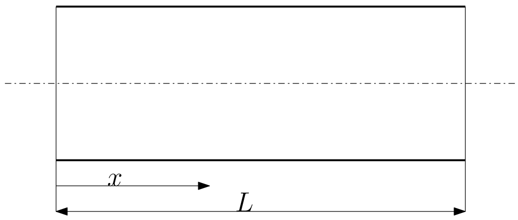


Fig. 3.3: Simple pipe of length L.

It is convenient to rewrite the equations (3.47) and (3.48) into the matrix formulation

$$\begin{bmatrix} 0 & \frac{\varrho}{S} \\ \frac{S}{\rho a^2} & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} p \\ Q \end{bmatrix} + \begin{bmatrix} 0 & \frac{b}{S} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} p \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.49}$$

denoting $\mathbf{w}(x,t) = \begin{bmatrix} p(x,t) & Q(x,t) \end{bmatrix}^{\mathrm{T}}$, we can write

$$\mathbf{K} \cdot \frac{\partial \mathbf{w}(x,t)}{\partial t} + \mathbf{B} \cdot \mathbf{w}(x,t) + \frac{\partial \mathbf{w}(x,t)}{\partial x} = 0 \tag{3.50}$$

Initial conditions determine the values of $\mathbf{w}(x,t)$ at time $t = 0$ in general at time $t = t_0$. They are usually given by the mean values of pressure p_0 and flow rate Q_0, thus

$$\mathbf{w} = \mathbf{w}(x,0) = \begin{bmatrix} p(x,0) & Q(x,0) \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} p_0 & Q_0 \end{bmatrix}^{\mathrm{T}}, \forall x \in \langle 0, L \rangle.$$

As (3.49) is the system of two partial differential equations we have to give one boundary condition for each end of the pipe. We use the notation from [9]. Because we can fix only one of the variables p, Q at the *LHS* end of the pipe, we denote the other one as unknown $\alpha(t)$. The prescribed functions for the tube end at $x = 0 (x = L)$ are in general the functions $\mathbf{m(t)}, n(t)$ of variable t respectively. Thus the boundary conditions can be expressed as

$$\mathbf{w}(0,t) = \mathbf{a_1}\alpha + \mathbf{a_2}\frac{\partial \alpha}{\partial t} + \mathbf{m}(t), \forall t \geq 0 \tag{3.51}$$

$$\mathbf{b_1}^{\mathrm{T}}\mathbf{w}(L,t) + \mathbf{b_2}^{\mathrm{T}}\frac{\partial \mathbf{w}}{\partial t}(L,t) = n(t), \forall t \geq 0, \tag{3.52}$$

where $\mathbf{a_1}, \mathbf{a_2}, \mathbf{b_1}, \mathbf{b_2}$ are known vectors according to the given boundary conditions.

3.2.2 Standard procedure for simple pipe

For the rest of this chapter, we will consider an undamped tube, thus the system of equations (3.49) is simplified as follows

$$\begin{bmatrix} 0 & \frac{\rho}{S} \\ \frac{S}{\rho a^2} & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} p \\ Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} p \\ Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{matrix} \forall x \in (0, L) \\ \forall t \geq 0 \end{matrix}. \tag{3.53}$$

The initial conditions are simple

$$\mathbf{w}(x, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}, \forall x \in \langle 0, L \rangle, \tag{3.54}$$

and boundary conditions

$$\mathbf{w}(0, t) = \mathbf{a_1} \alpha(t) + \mathbf{m}(t), \forall t \geq 0 \tag{3.55}$$

$$\mathbf{b_1}^{\mathrm{T}} \mathbf{w}(L, t) = n(t), \forall t \geq 0, \tag{3.56}$$

where

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \alpha = p(0, t), \ \mathbf{m}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} h(t), \mathbf{b_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q(L, t) = n(t) = 0. \tag{3.57}$$

To solve this problem we will use the double *Laplace transform*, thus the *Laplace images* of terms in (3.57)

$$\begin{aligned} \mathscr{L}\{\mathbf{w}(x, t)\}_{t \to s} &= \mathbf{u}(x, s) & \mathscr{L}\{\alpha(0, t)\}_{t \to s} &= \beta(0, s) \\ \mathscr{L}\{\mathbf{m}(t)\}_{t \to s} &= \mathbf{r}(0, s) & \mathscr{L}\{n(t)\}_{t \to s} &= z(s) \end{aligned} \tag{3.58}$$

Applying the Laplace transform $\mathscr{L}\{\cdot\}_{x,t \to \varepsilon,s}$ to the equation (3.53) gives

$$\begin{aligned} &\begin{bmatrix} 0 & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & 0 \end{bmatrix} \cdot \begin{bmatrix} \{p(x, t)\}_{x,t \to \varepsilon,s} \\ \{Q(x, t)\}_{x,t \to \varepsilon,s} \end{bmatrix} - \begin{bmatrix} \{p(x, 0)\}_{x \to \varepsilon} \\ \{Q(x, 0)\}_{x \to \varepsilon} \end{bmatrix} + \\ &\qquad \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} \{p(x, t)\}_{x,t \to \varepsilon,s} \\ \{Q(x, t)\}_{x,t \to \varepsilon,s} \end{bmatrix} - \begin{bmatrix} \{p(0, t)\}_{t \to s} \\ \{Q(0, t)\}_{t \to s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{3.59}$$

considering the *Laplace images* of boundary conditions (3.58) and initial conditions given in (3.54) gives

$$\begin{bmatrix} \varepsilon & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} \{p(x, t)\}_{x,t \to \varepsilon,s} \\ \{Q(x, t)\}_{x,t \to \varepsilon,s} \end{bmatrix} = \begin{bmatrix} \{p(0, t)\}_{t \to s} \\ r(0, s) \end{bmatrix}. \tag{3.60}$$

Rearranging the previous equation gives

$$\begin{bmatrix} \{p(x, t)\}_{x,t \to \varepsilon,s} \\ \{Q(x, t)\}_{x,t \to \varepsilon,s} \end{bmatrix} = \begin{bmatrix} \varepsilon & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & \varepsilon \end{bmatrix}^{-1} \cdot \begin{bmatrix} \{p(0, t)\}_{t \to s} \\ r(0, s) \end{bmatrix}. \tag{3.61}$$

We denote the inverse matrix as $\mathbf{P}$

$$\mathbf{P}(\varepsilon, s) = \begin{bmatrix} \varepsilon & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & \varepsilon \end{bmatrix}^{-1} = \frac{1}{\varepsilon^2 - \frac{s^2}{a^2}} \begin{bmatrix} \varepsilon & -\frac{s\rho}{S} \\ -\frac{sS}{\rho a^2} & \varepsilon \end{bmatrix}, \tag{3.62}$$

and the *inverse Laplace transform* $\mathscr{L}^{-1}\{\mathbf{P}\}_{x \to \varepsilon}$ gives

$$\mathbf{u}(x, s) = \begin{bmatrix} \{p(x, t)\}_{t \to s} \\ \{Q(x, t)\}_{t \to s} \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a} x\right) & \frac{-\rho a}{S} \sinh\left(\frac{s}{a} x\right) \\ -\frac{S}{\rho a} \sinh\left(\frac{s}{a} x\right) & \cosh\left(\frac{s}{a} x\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0, t)\}_{t \to s} \\ r(0, s) \end{bmatrix}. \tag{3.63}$$

The unknown term $\{p(0,t)\}_{t\to s}$ from in the equation (3.63) will be determined by using the boundary condition (3.56) for the *RHS* end of the beam.

$$\begin{bmatrix} \{p(L,t)\}_{t\to s} \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a}L\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}L\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}L\right) & \cosh\left(\frac{s}{a}L\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ r(0,s) \end{bmatrix}, \tag{3.64}$$

The second equation in (3.64) gives the equality

$$-\frac{S}{\rho a}\sinh\left(\frac{s}{a}L\right)\{p(0,t)\}_{t\to s} + \cosh\left(\frac{s}{a}L\right)r(0,s) = 0.$$

As we express the term $\{p(0,t)\}_{t\to s}$ and substitute it into the equation (3.63) we get

$$\mathbf{u}(x,s) = \begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a}x\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}x\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}x\right) & \cosh\left(\frac{s}{a}x\right) \end{bmatrix} \cdot \begin{bmatrix} \frac{\rho a}{S}r(s)\coth(\frac{s}{a}L) \\ r(0,s) \end{bmatrix}. \tag{3.65}$$

It is possible to express (3.65) as

$$\mathbf{u}(x,s) = \mathbf{P}(x,s)\cdot\mathbf{u}(0,s). \tag{3.66}$$

Finally, we have *Laplace images* of functions for pressure and flow rate, which depend only on the *Laplace image* of the known function $\mathscr{L}\{h(0,t)\}_{t\to s} = r(s)$.

$$\{p(x,t)\}_{t\to s} = r(0,s)\frac{\rho a}{S}\cosh\left(\frac{s}{a}x\right)\coth\left(\frac{s}{a}L\right) - r(0,s)\frac{\rho a}{S}\sinh\left(\frac{s}{a}x\right) \tag{3.67}$$

$$\{Q(x,t)\}_{t\to s} = -r(0,s)\sinh\left(\frac{s}{a}x\right)\coth\left(\frac{s}{a}L\right) + r(0,s)\cosh\left(\frac{s}{a}x\right) \tag{3.68}$$

To obtain the originals of the equations (3.67) and (3.68) in the variable t, we have to perform an *inverse Laplace transform* according to the Theorem 3.

3.2.3 General piping system

In the previous subchapter, we solved the problem for the simple pipe, so there is an obvious question, how to extend those results.

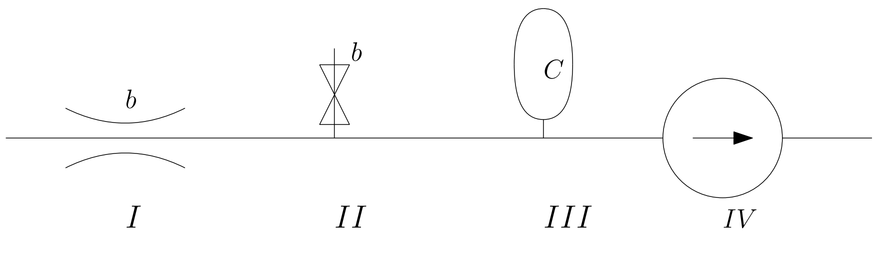


Fig. 3.4: General piping system. Redrawed from [9].

As an extension, we will consider the serial connection of simple pipes, which will be joint by the local hydraulic elements as shown in the Figure 3.4. The local hydraulic elements I, II, III, IV are described by their characteristic constants.

Look at the local hydraulic element I, called *hydraulic resistance*, for which the following relations were derived in [9]

$$p_{i+1}(0,t) = p_i(L_i,t) - bQ_i(L_i,t), \tag{3.69}$$
$$Q_{i+1}(0,t) = Q_i(L_i,t). \tag{3.70}$$

From the equations (3.69) and (3.70) it can be seen that pressure drop at point $x = 0$ in the pipe with index $i+1$ after the *hydraulic resistance*, with characteristic b, depends on the pressure $p_i(L,t)$ and flow rate $Q_i(L,t)$ at the right end $x = L_i$ of the pipe i.

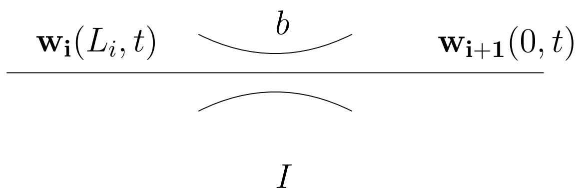


Fig. 3.5: Hydraulic resistance I. Redrawed from [9].

Rewriting the equations (3.69) and (3.70) gives the matrix identity

$$\underbrace{\begin{bmatrix} p_{i+1}(0,t) \\ Q_{i+1}(0,t) \end{bmatrix}}_{\mathbf{w_{i+1}}(0,t)} = \underbrace{\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}}_{\mathbf{R_i}} \cdot \underbrace{\begin{bmatrix} p_i(L,t) \\ Q_i(L,t) \end{bmatrix}}_{\mathbf{w_i}(L_i,t)}. \tag{3.71}$$

Laplace transform of the equation (3.71) can be simply written as

$$\mathbf{u_{i+1}}(0,s) = \mathbf{R_i} \cdot \mathbf{u_i}(L_i,s). \tag{3.72}$$

We deal with an N-pipe piping system with the *hydraulic elements* listed in the Table 3.1, zero initial conditions as defined in (3.54) and boundary conditions from (3.55) and (3.56). Considering the equation (3.66), $\mathbf{u_i}(x,s) = \mathbf{P_i}(x,s) \cdot \mathbf{u_i}(0,s)$, which shows that *Laplace image* of the vector $\mathscr{L}\{\mathbf{w}(\mathbf{x},\mathbf{t})\}_{t \to s} = \mathbf{u}(x,s)$ at each position $x \in \langle 0, L_i \rangle$ is uniquely determined by the transfer matrix $\mathbf{P_i}$ and an initial vector $\mathbf{u_i}(0,s)$. The identity (3.72) provides the possibility to move from the *RHS* end of tube i to the *LHS* end of the pipe $i+1$, so we can easily write down the system of equations

$$\begin{array}{ccccc}
\mathbf{a}\beta & - & \mathbb{I}\mathbf{u_1} & = & -\mathbf{x_0} \\
\mathbf{R_1 P_1 u_1} & - & \mathbb{I}\mathbf{u_2} & = & -\mathbf{x_1} \\
\mathbf{R_2 P_2 u_2} & - & \mathbb{I}\mathbf{u_3} & = & -\mathbf{x_2} \\
\vdots & & \vdots & & \vdots \\
\mathbf{R_i P_i u_i} & - & \mathbb{I}\mathbf{u_{i+1}} & = & -\mathbf{x_i} \\
\vdots & & \vdots & & \vdots \\
\mathbf{R_{N-1} P_{N-1} u_{N-1}} & - & \mathbb{I}\mathbf{u_N} & = & -\mathbf{x_{N-1}} \\
\mathbf{b}^{\mathrm{T}} \mathbf{P_N u_N} & & & = & -x_N,
\end{array} \tag{3.73}$$

which can be rewritten into matrix form as

$$
\left[\begin{array}{c|ccccccc}
\mathbf{a} & -\mathbb{I} & & & & & & \\
 & \mathbf{R_1P_1} & -\mathbb{I} & & & & & \\
 & & \mathbf{R_2P_2} & -\mathbb{I} & & & & \\
 & & & \ddots & \ddots & & & \\
 & & & & \mathbf{R_iP_i} & -\mathbb{I} & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & \mathbf{R_{N-1}P_{N-1}} & -\mathbb{I} \\
\hline
 & & & & & & & \mathbf{b}^{\mathrm{T}}\mathbf{P_N}
\end{array}\right]
\begin{bmatrix} \beta \\ \mathbf{u_1} \\ \mathbf{u_2} \\ \vdots \\ \mathbf{u_i} \\ \vdots \\ \mathbf{u_{N-1}} \\ \mathbf{u_N} \end{bmatrix}
= -
\begin{bmatrix} \mathbf{x_0} \\ \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_i} \\ \vdots \\ \mathbf{x_{N-1}} \\ x_N \end{bmatrix},
\tag{3.74}
$$

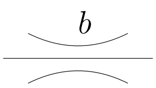
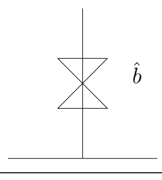
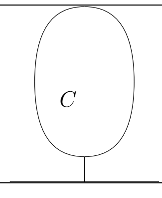
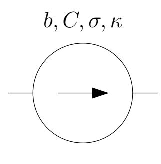
where

$$
\begin{array}{rcccrcl}
\mathbf{x_0} & = & \mathbf{r}(s) - \mathbf{a_2}\,\alpha(0,0) & \Rightarrow & \mathbf{a}\beta - \mathbf{u}_1 & = & -\mathbf{x_0} \\
x_n & = & -z(s) - \mathbf{b}^{\mathrm{T}}\mathbf{w}(L_N,0) & \Rightarrow & \mathbf{b}^{\mathrm{T}}\mathbf{P_N}\mathbf{u_N} & = & -x_N \\
\mathbf{x_i} & = & \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}, \forall i = 1,2,\ldots,N-1 & & & & \\
\mathbf{u_i} & = & \mathbf{u_i}(L_i,s) & & & & \\
\mathbf{P_i} & = & \mathbf{P_i}(L_i,s) & & & &
\end{array}
$$

Solution of the matrix equation (3.74) gives the boundary conditions at *LHS* end of the each pipe, i.e. vector $\mathbf{u_i}(0,s), \forall i = 1,2,\ldots,N$. According to the equation (3.66) we can express vector $\mathbf{u_i}(x,s) = \mathbf{P_i}(x,s) \cdot \mathbf{u_i}(0,s), \forall i = 1,2,\ldots,N$ and by the *inverse Laplace transform* obtain the function $\mathbf{w_i}(x,t)$ in the variables x,t for each pipe i.

Note. We consider the hydraulic elements shown in the Figure 3.4. The transition matrix $\mathbf{R_i}$, together with the schematic sign of each of the *hydraulic element* is in the Table 3.1

Tab. 3.1: Hydraulic elements with their transition matrices

Name	Hydraulic resistance	Parallel connected hydraulic resistance	Gas accumulator without damping	Centrifugal pump
Schematic sign				
$\mathbf{R_i}$	$\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -\frac{1}{\hat{b}} & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -sC & 1 \end{bmatrix}$	$\begin{bmatrix} 1+\sigma & -b \\ -sC & 1-s\kappa \end{bmatrix}$

3.2.4 Piping system with *hydraulic resistance*

We consider a simple 2-pipe piping system, where both pipes have the same length L and characteristic parameters. A local hydraulic element, from the Table 3.1, mediates the connection of the pipes. In the first case, we will use a *hydraulic resistance* with parameter b, which has to be determined experimentally. The considered problem is shown in the Figure 3.6.

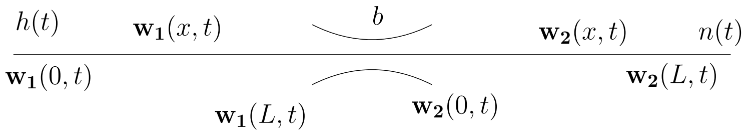


Fig. 3.6: Simple piping system, with *hydraulic resistance*.

Given initial conditions are $\mathbf{w_i}(x,0) = \begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}, \forall x \in \langle 0, L_i \rangle, i = 1,2$, moreover we have the same boundary conditions as in the example in the subsection 3.2.2. The matrix equation (3.74) for this piping system can be written as[5]

$$\underbrace{\begin{bmatrix} \mathbf{a} & -\mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R_1 P_1} & -\mathbb{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{b}^{\mathrm{T}}\mathbf{P_2} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} \beta \\ \mathbf{u_1} \\ \mathbf{u_2} \end{bmatrix}}_{\mathbf{Y}} = \underbrace{\begin{bmatrix} \mathbf{r(s)} \\ \mathbf{x_1} \\ x_2 \end{bmatrix}}_{\mathbf{x}}, \tag{3.75}$$

where the *transition matrix* for the *hydraulic resistance* from the Table 3.1 is

$$\mathbf{R_1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}. \tag{3.76}$$

For both characteristic matrices $\mathbf{P_i}, i = 1,2$, which were introduced in (3.62), we have

$$\mathbf{P_1} = \mathbf{P_2} = \mathbf{P}(x,s) = \begin{bmatrix} \cosh\left(\frac{sx}{a}\right) & -\frac{a\rho}{S}\sinh\left(\frac{sx}{a}\right) \\ -\frac{S}{a\rho}\sinh\left(\frac{sx}{a}\right) & \cosh\left(\frac{sx}{a}\right) \end{bmatrix}. \tag{3.77}$$

Substituting (3.76) and (3.77) into the matrix $\mathbf{A}$ in (3.75) gives

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right) + \cosh\left(\frac{Ls}{a}\right) & -b\cosh\left(\frac{Ls}{a}\right) - \frac{a\rho}{S}\sinh\left(\frac{Ls}{a}\right) & -1 & 0 \\ 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) & 0 & -1 \\ 0 & 0 & 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.78}$$

[5]Symbol $\mathbf{0}$ is the zero matrix of appropriate dimension.

The *RHS* in (3.75) is in a simple form $\mathbf{x} = \begin{bmatrix} 0 & -r & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. The elements of the unknown matrix of the coefficients in (3.75), $\mathbf{Y} = \begin{bmatrix} \beta & p_1(0,s) & Q_1(0,s) & p_2(0,s) & Q_2(0,s) \end{bmatrix}^{\mathrm{T}}$, represent the boundary conditions for the *LHS* end of the pipes $\mathbf{u_1}(0,s)$ and $\mathbf{u_2}(0,s)$.

$$\begin{bmatrix} \beta \\ p_1(0,s) \\ Q_1(0,s) \\ p_2(0,s) \\ Q_2(0,s) \end{bmatrix} = \begin{bmatrix} \frac{ar\rho\left(Sb\sinh\left(\frac{2L}{a}s\right)+2a\rho\cosh\left(\frac{2L}{a}s\right)\right)}{S\left(Sb\cosh\left(\frac{2L}{a}s\right)-Sb+2a\rho\sinh\left(\frac{2L}{a}s\right)\right)} \\ \frac{ar\rho\left(Sb\sinh\left(\frac{2L}{a}s\right)+2a\rho\cosh\left(\frac{2L}{a}s\right)\right)}{S\left(Sb\cosh\left(\frac{2L}{a}s\right)-Sb+2a\rho\sinh\left(\frac{2L}{a}s\right)\right)} \\ r \\ \frac{a^2r\rho^2}{S\left(Sb\tanh\left(\frac{Ls}{a}\right)+2a\rho\right)\sinh\left(\frac{Ls}{a}\right)} \\ \frac{ar\rho}{Sb\sinh\left(\frac{Ls}{a}\right)+2a\rho\cosh\left(\frac{Ls}{a}\right)} \end{bmatrix}. \tag{3.79}$$

The solution (3.79) allows us to express a *Laplace image* of the state functions $p_i(x,s)$, $Q_i(x,s), i = 1,2$ in any positions x of the two pipes, using the identity (3.66).

3.2.5 Piping system with *gas accumulator without damping*

Let us have a look at the similar problem as we dealt with in the subsection 3.2.4. We have a simple 2-pipe piping system, but the *local hydraulic* term is the *gas accumulator without damping*, where the characteristic coefficient $C = \frac{V}{\rho v^2}$, describes the isothermal change of the volume V of the barotropic fluid. For the isothermal change of ideal gas with the mean volume V_0 and pressure p_0 in the accumulator we have $C = \frac{V_0}{p_0}$. The situation in this second example for simple piping system is in the Figure 3.7.

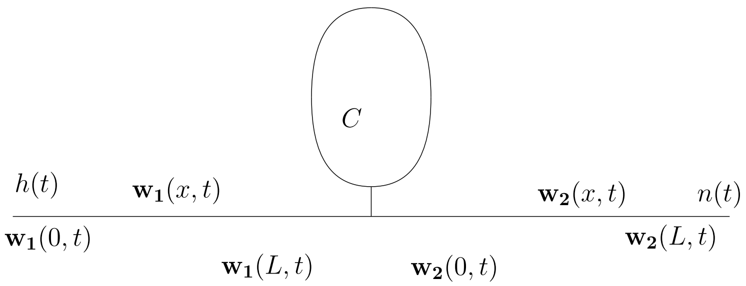


Fig. 3.7: Simple piping system, with *Gas accumulator without damping.*

System of equations (3.74) for this piping system is the same as in the previous example, but the *transition matrix* is different, according to the Table 3.1 we have

$$\mathbf{R_1} = \begin{bmatrix} 1 & 0 \\ -sC & 1 \end{bmatrix}. \tag{3.80}$$

Matrix $\mathbf{A}$ from (3.74) for this system with the *transition matrix* (3.80) is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \cosh\left(\frac{Ls}{a}\right) & -\frac{a\rho}{S}\sinh\left(\frac{Ls}{a}\right) & -1 & 0 \\ 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) - Cs\cosh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) + \frac{aC}{S}\rho s\sinh\left(\frac{Ls}{a}\right) & 0 & -1 \\ 0 & 0 & 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.81}$$

System of equations (3.74) can be easily solved as the *RHS* is the same as in the previous example. The resulting vector enables us to determine the *Laplace image* of the state vector $\mathbf{u_i}(x, s), i = 1, 2$.

3.2.6 Piping system with *damped gas accumulator*

As we have solved two problems of piping systems, where *hydraulic terms* were *hydraulic resistance* and *gas accumulator* respectively, consider a joining of these two elements together to obtain a *damped gas accumulator*. We can characterise this element with two coefficients as in the previous subsections: b, C. The situation in this example is in the Figure 3.8.

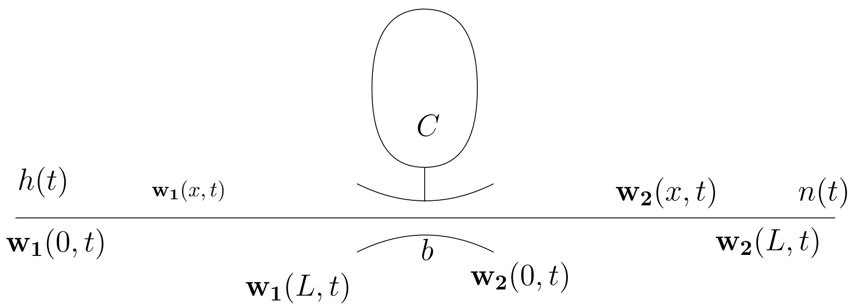


Fig. 3.8: Simple piping system, with *damped gas accumulator*.

The procedures of solving this system are the same as in the previous subsections 3.2.4 and 3.2.5, thus we only state the matrices $\mathbf{R_1}, \mathbf{A}$ and the resulting matrix of the unknown coefficients.

$$\mathbf{R_1} = \begin{bmatrix} 1 & -b \\ -sC & 1 \end{bmatrix}. \tag{3.82}$$

Substituting (3.82) into the matrix equation (3.74) gives the matrix $\mathbf{A}$ of the system

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right) + \cosh\left(\frac{Ls}{a}\right) & -b\cosh\left(\frac{Ls}{a}\right) - \frac{a\rho}{S}\sinh\left(\frac{Ls}{a}\right) & -1 & 0 \\ 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) - Cs\cosh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) + \frac{aC}{S}\rho s\sinh\left(\frac{Ls}{a}\right) & 0 & -1 \\ 0 & 0 & 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.83}$$

Solving the system of equations (3.74) with matrix $\mathbf{A}$ given by (3.83) and the matrix of the *RHS* $\mathbf{x} = \begin{bmatrix} 0 & -r & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ results in the wanted vector of unknowns which we substitute back into (3.66) and obtain the *Laplace images* of the state functions

$p_i(x,s), Q_i(x,s)$ in both pipes.

$$\begin{bmatrix} \beta \\ p_1(0,s) \\ Q_1(0,s) \\ p_2(0,s) \\ Q_2(0,s) \end{bmatrix} = \begin{bmatrix} \frac{ar\rho\left(S^2b\sinh\left(\frac{2L}{a}s\right)+2Sa\rho\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\sinh\left(\frac{2L}{a}s\right)\right)}{S\left(S^2b\cosh\left(\frac{2L}{a}s\right)-S^2b+2Sa\rho\sinh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\right)} \\ \frac{ar\rho\left(S^2b\sinh\left(\frac{2L}{a}s\right)+2Sa\rho\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\sinh\left(\frac{2L}{a}s\right)\right)}{S\left(S^2b\cosh\left(\frac{2L}{a}s\right)-S^2b+2Sa\rho\sinh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\right)} \\ r \\ -\frac{2a^2r\rho^2(bCs-1)\cosh\left(\frac{Ls}{a}\right)}{S^2b\cosh\left(\frac{2L}{a}s\right)-S^2b+2Sa\rho\sinh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s} \\ -\frac{2Sar\rho(bCs-1)\sinh\left(\frac{Ls}{a}\right)}{S^2b\cosh\left(\frac{2L}{a}s\right)-S^2b+2Sa\rho\sinh\left(\frac{2L}{a}s\right)+a^2C\rho^2s\cosh\left(\frac{2L}{a}s\right)+a^2C\rho^2s} \end{bmatrix}.$$

3.2.7 Piping system with *parallel hydraulic resistance*

A simple 2-pipe piping system, with *hydraulic resistance* connected in a parallel way into the system is shown in Figure 3.9. The *hydraulic term* is characterized by the coefficient $\hat{b}$, which is usually determined experimentally. This problem can be easily solved as in

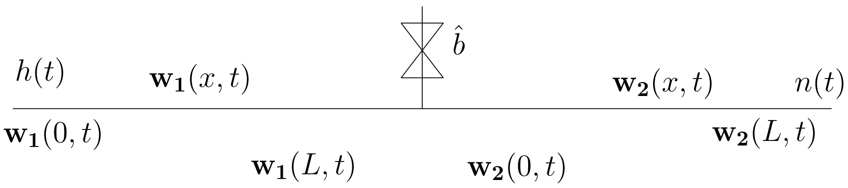


Fig. 3.9: Simple piping system, with *parallel hydraulic resistance*.

the previous example as we substitute *transition matrix* defined for this *hydraulic term*

$$\mathbf{R_1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\hat{b}} & 1 \end{bmatrix},$$

into the system of equations (3.74)

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \cosh\left(\frac{Ls}{a}\right) & -\frac{a\rho}{S}\sinh\left(\frac{Ls}{a}\right) & -1 & 0 \\ 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right)-\frac{1}{\hat{b}}\cosh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right)+\frac{a\rho}{S\hat{b}}\sinh\left(\frac{Ls}{a}\right) & 0 & -1 \\ 0 & 0 & 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) \end{bmatrix} \tag{3.84}$$

3.2.8 Piping system with *centrifugal pump*

Last example of the *hydraulic terms* from the Table 3.1 has the most complicated *transient matrix*

$$\mathbf{R_1} = \begin{bmatrix} 1+\sigma & -b \\ -sC & 1-s\kappa \end{bmatrix}, \tag{3.85}$$

where b, C, κ, σ characterise the *centrifugal pump*. Damping factor b can be positive or negative, as the first option is found at the pumps with the stable characteristics, the other one at the non-stable ones. Coefficients σ, κ describes the conditions, when the

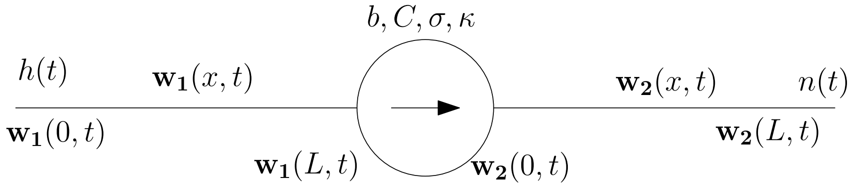


Fig. 3.10: Simple piping system, with *centrifugal pump.*

pump is not in the optimal setting, and the cavitation effect can be observed. This system is in the Figure 3.10. The matrix $\mathbf{A}$ from (3.74) in this case is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right) + (\sigma+1)\cosh\left(\frac{Ls}{a}\right) & -b\cosh\left(\frac{Ls}{a}\right) - \frac{a\rho}{S}(\sigma+1)\sinh\left(\frac{Ls}{a}\right) & -1 & 0 \\ 0 & -\frac{S}{a\rho}(-\kappa s+1)\sinh\left(\frac{Ls}{a}\right) - cs\cosh\left(\frac{Ls}{a}\right) & (-\kappa s+1)\cosh\left(\frac{Ls}{a}\right) + \frac{ac}{S}\rho s\sinh\left(\frac{Ls}{a}\right) & 0 & -1 \\ 0 & 0 & 0 & -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & \cosh\left(\frac{Ls}{a}\right) \end{bmatrix} \tag{3.86}$$

3.3 Generalized function approach

In this subsection, we focus on solving the problems mentioned in the previous subsections 3.2.4 -3.2.8, but we will approach the solution through distributions. The equations (3.47) and (3.48) for the state variables $p(x,t), Q(x,t)$ with the *hydraulic terms* can be rewritten using the *Dirac delta* distribution as

$$\frac{\partial p(x,t)}{\partial x} + \frac{\rho}{S}\frac{\partial Q(x,t)}{\partial t} - \sigma\, p(\zeta,t)\,\delta(x-\zeta) + b\, Q(\zeta,t)\,\delta(x-\zeta) = 0 \tag{3.87}$$

$$\frac{\partial Q(x,t)}{\partial x} + \frac{S}{\rho a^2}\frac{\partial p(x,t)}{\partial t} + C\frac{\partial p(\zeta,t)}{\partial t}\,\delta(x-\zeta) + \kappa\,\frac{\partial Q(\zeta,t)}{\partial t}\,\delta(x-\zeta) = 0. \tag{3.88}$$

It is convenient to rewrite (3.87) and (3.88) into the matrix equation

$$\begin{bmatrix} 0 & \frac{\rho}{S} \\ \frac{S}{\rho a^2} & 0 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} p \\ Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} p \\ Q \end{bmatrix} = \begin{bmatrix} \sigma\, p(\zeta,t)\,\delta(x-\zeta) - b\, Q(\zeta,t)\,\delta(x-\zeta) \\ C\,\frac{\partial p(\zeta,t)}{\partial t}\,\delta(x-\zeta) - \kappa\,\frac{\partial Q(\zeta,t)}{\partial t}\,\delta(x-\zeta) \end{bmatrix}. \tag{3.89}$$

Applying the *Laplace transform* $\mathscr{L}\{\cdot\}_{t\to s}$ to the equation (3.89), with zero initial condition $\mathbf{w}(x,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ gives

$$\begin{bmatrix} 0 & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & 0 \end{bmatrix}\begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} = \\ \begin{bmatrix} \sigma\,\{p(\zeta,t)\}_{t\to s}\delta(x-\zeta) - b\,\{Q(\zeta,t)\}_{t\to s}\,\delta(x-\zeta) \\ -s\,C\{p(\zeta,t)\}_{t\to s}\,\delta(x-\zeta) - s\kappa\{Q(\zeta,t)\}_{t\to s}\delta(x-\zeta) \end{bmatrix}. \tag{3.90}$$

To obtain the *double Laplace transform* $\mathscr{L}\{\cdot\}_{x,t\to\varepsilon,s}$ of (3.89), we have to perform the second transform $\mathscr{L}\{\cdot\}_{x\to\varepsilon}$ to the equation (3.90)

$$\begin{bmatrix} \varepsilon & \frac{s\rho}{S} \\ \frac{sS}{\rho a^2} & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} \{p(x,t)\}_{x,t\to\varepsilon,s} \\ \{Q(x,t)\}_{x,t\to\varepsilon,s} \end{bmatrix} = \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{Q(0,t)\}_{t\to s} \end{bmatrix} + \begin{bmatrix} \sigma\{p(\zeta,t)\}_{t\to s}\,\mathrm{e}^{-\zeta\varepsilon} - b\{Q(\zeta,t)\}_{t\to s}\,\mathrm{e}^{-\zeta\varepsilon} \\ -sC\{p(\zeta,t)\}_{t\to s}\,\mathrm{e}^{-\zeta\varepsilon} - s\kappa\{Q(\zeta,t)\}_{t\to s}\,\mathrm{e}^{-\zeta\varepsilon} \end{bmatrix}. \quad (3.91)$$

We express the matrix $\mathbf{P}(x,s)$ as in the subsection 3.2.2, but the *inverse Laplace transform* is different due to the term $\mathrm{e}^{-\zeta\varepsilon}$, thus we obtain

$$\begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a}x\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}x\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}x\right) & \cosh\left(\frac{s}{a}x\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{Q(0,t)\}_{t\to s} \end{bmatrix} +$$
$$\mathrm{H}(x-\zeta) \begin{bmatrix} \cosh\left(\frac{s}{a}(x-\zeta)\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}(x-\zeta)\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(x-\zeta)\right) & \cosh\left(\frac{s}{a}(x-\zeta)\right) \end{bmatrix} \cdot \begin{bmatrix} \sigma\{p(\zeta,t)\}_{t\to s} - b\{Q(\zeta,t)\}_{t\to s} \\ -sC\{p(\zeta,t)\}_{t\to s} - s\kappa\{Q(\zeta,t)\}_{t\to s} \end{bmatrix} \quad (3.92)$$

3.3.1 Piping system with ***hydraulic resistance***

We have the same problem as in the subsection 3.2.4

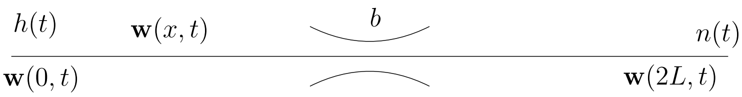


Fig. 3.11: Simple piping system, with *hydraulic resistance*.

System of equations (3.92) for this particular case, when $C = \kappa = \sigma = 0$, due to the fact that there is no *gas accumulator without damping* nor *centrifugal pump*, and $\zeta = L$ is

$$\begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a}x\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}x\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}x\right) & \cosh\left(\frac{s}{a}x\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ r(s) \end{bmatrix} +$$
$$\mathrm{H}(x-L) \begin{bmatrix} \cosh\left(\frac{s}{a}(x-L)\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}(x-L)\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(x-L)\right) & \cosh\left(\frac{s}{a}(x-L)\right) \end{bmatrix} \cdot \begin{bmatrix} -b\{Q(L,t)\}_{t\to s} \\ 0 \end{bmatrix} \quad (3.93)$$

The equation (3.93) is somehow similar to an expression (3.66), but in the equation (3.93) there are two unknown coefficients $\{p(0,t)\}_{t\to s}$ and $\{Q(L,t)\}_{t\to s}$, in order to express these terms we use the operator defined in Definition 7[6]. Applying the operators defined as

$$\mathcal{E}_{2L} = \begin{bmatrix} 0 \\ \mathcal{E}_{2L} \end{bmatrix} \text{ and } \mathcal{E}_L = \begin{bmatrix} 0 \\ \mathcal{E}_L \end{bmatrix},$$

[6] To apply the operators mentioned above, it is necessary to make the *Laplace transform* of equation (3.93) formally and its inversion consequently.

gives the system of two equations for two unknown coefficients

$$\begin{bmatrix} \{Q(2L,t)\}_{t\to s} \\ \{Q(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -\frac{S}{\rho a}\sinh\left(\frac{s}{a}2L\right) & \cosh\left(\frac{s}{a}2L\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}L\right) & \cosh\left(\frac{s}{a}L\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ r(s) \end{bmatrix} +$$
$$\begin{bmatrix} -\mathrm{H}(2L-L)\frac{S}{\rho a}\sinh\left(\frac{s}{a}(2L-L)\right) & \mathrm{H}(2L-L)\cosh\left(\frac{s}{a}(2L-L)\right) \\ -\mathrm{H}(L-L)\frac{S}{\rho a}\sinh\left(\frac{s}{a}(L-L)\right) & \mathrm{H}(L-L)\cosh\left(\frac{s}{a}(L-L)\right) \end{bmatrix} \cdot \begin{bmatrix} -b\,\{Q(L,t)\}_{t\to s} \\ 0 \end{bmatrix}. \tag{3.94}$$

The simplification of (3.94) results in the simple matrix equation

$$\begin{bmatrix} -\frac{S}{a\rho}\sinh\left(\frac{2L}{a}s\right) & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right) \\ \frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & 1 \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{Q(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -r\cosh\left(\frac{2L}{a}s\right) \\ r\cosh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.95}$$

Solving the matrix equation (3.95) gives us the unknown coefficients

$$\begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{Q(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \frac{ar\rho\left(Sb\sinh\left(\frac{2L}{a}s\right)+2a\rho\cosh\left(\frac{2L}{a}s\right)\right)}{S\left(Sb\cosh\left(\frac{2L}{a}s\right)-Sb+2a\rho\sinh\left(\frac{2L}{a}s\right)\right)} \\ \frac{ar\rho}{Sb\sinh\left(\frac{Ls}{a}\right)+2a\rho\cosh\left(\frac{Ls}{a}\right)} \end{bmatrix}, \tag{3.96}$$

which we will substitute back into (3.93) and obtain the *Laplace image* of the state vector $\mathscr{L}\{\mathbf{w}(x,t)\}_{t\to s} = \mathbf{u}(x,s), \forall x \in \langle 0, 2L\rangle$.

3.3.2 Piping system with *gas accumulator without damping*

Let us consider the piping system from the subsection 3.2.5, which was solved as two depending pipes. We will approach this problem, pictured in Figure 3.12, using the methods defined in 3.3.

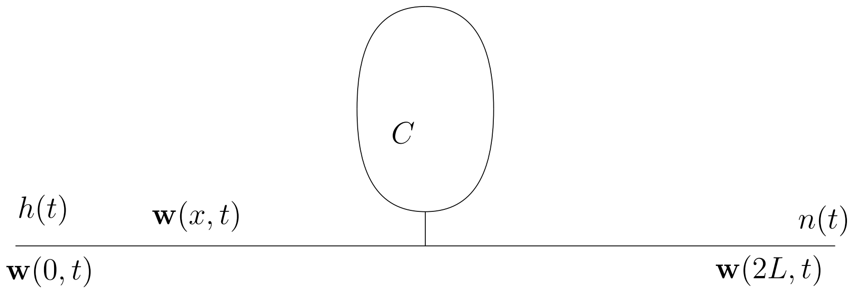


Fig. 3.12: Simple piping system, with *gas accumulator without damping*.

The system of equations (3.92) is modified because we have no *hydraulic resistance, centrifugal pump* and the pipes are joint together by the element called *gas accumulator without damping*. The assumptions $b = \kappa = \sigma = 0$ and $\zeta = L$ implies

$$\begin{bmatrix} \{p(x,t)\}_{t\to s} \\ \{Q(x,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \cosh\left(\frac{s}{a}x\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}x\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}x\right) & \cosh\left(\frac{s}{a}x\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ r(s) \end{bmatrix} +$$
$$\mathrm{H}(x-L) \begin{bmatrix} \cosh\left(\frac{s}{a}(x-L)\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}(x-L)\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(x-L)\right) & \cosh\left(\frac{s}{a}(x-L)\right) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -sC\,\{p(L,t)\}_{t\to s} \end{bmatrix} \tag{3.97}$$

In the equation (3.97) there are two unknown coefficients $\{p(0,t)\}_{t\to s}$ and $\{p(L,t)\}_{t\to s}$. We use previously defined operators

$$\mathcal{E}_{2L} = \begin{bmatrix} 0 \\ \mathcal{E}_{2L} \end{bmatrix} \text{ and } \mathcal{E}_{L} = \begin{bmatrix} \mathcal{E}_{L} \\ 0 \end{bmatrix},$$

formally applied to the system (3.97) to obtain matrix equation

$$\begin{bmatrix} \{Q(2L,t)\}_{t\to s} \\ \{p(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -\frac{S}{\rho a}\sinh\left(\frac{s}{a}2L\right) & \cosh\left(\frac{s}{a}2L\right) \\ \cosh\left(\frac{s}{a}L\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}L\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ r(s) \end{bmatrix} +$$
$$\begin{bmatrix} -\mathrm{H}(2L-L)\frac{S}{\rho a}\sinh\left(\frac{s}{a}(2L-L)\right) & \mathrm{H}(2L-L)\cosh\left(\frac{s}{a}(2L-L)\right) \\ \mathrm{H}(L-L)\cosh\left(\frac{s}{a}(L-L)\right) & \mathrm{H}(L-L)\frac{-\rho a}{S}\sinh\left(\frac{s}{a}(L-L)\right) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -sC\,\{p(L,t)\}_{t\to s} \end{bmatrix}. \tag{3.98}$$

Transcription of (3.98) simplifies the problem

$$\begin{bmatrix} -\frac{S}{a\rho}\sinh\left(\frac{2L}{a}s\right) & -sC\,\cosh\left(\frac{Ls}{a}\right) \\ \cosh\left(\frac{Ls}{a}\right) & -1 \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{p(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -r\cosh\left(\frac{2L}{a}s\right) \\ \frac{ar}{S}\rho\sinh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.99}$$

Unknown coefficients are obtained as a solution of the matrix equation (3.99)

$$\begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} \frac{ar\rho\left(2S\cosh\left(\frac{2L}{a}s\right)+aC\rho s\sinh\left(\frac{2L}{a}s\right)\right)}{S\left(2S\sinh\left(\frac{2L}{a}s\right)+aC\rho s\cosh\left(\frac{2L}{a}s\right)+aC\rho s\right)} \\ \frac{ar\rho\cosh\left(\frac{Ls}{a}\right)}{S\sinh\left(\frac{2L}{a}s\right)+\frac{aC}{2}\rho s\cosh\left(\frac{2L}{a}s\right)+\frac{aC}{2}\rho s} \end{bmatrix}, \tag{3.100}$$

and substituting these unknowns back into (3.97) gives the *Laplace images* of the state variables $p(x,s), Q(x,s), \forall x \in \langle 0, 2L\rangle$.

3.3.3 Piping system with *damped gas accumulator*

Similarly to the subsection 3.2.6, we can combine the two *hydraulic elements* from the subsections 3.3.1 and 3.3.2 and obtain new element denoted as *damped gas accumulator*. The situation in this example is in the Figure 3.13. Due to the fact that both previously

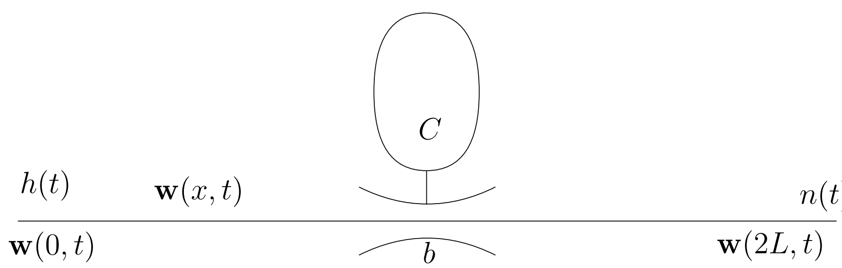


Fig. 3.13: Simple piping system, with *damped gas accumulator*.

encountered *hydraulic elements* are applied at the position $x = L$, we modify the system of equations (3.92) into

$$\begin{bmatrix}\{p(x,t)\}_{t\to s}\\ \{Q(x,t)\}_{t\to s}\end{bmatrix} = \begin{bmatrix}\cosh\left(\frac{s}{a}x\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}x\right)\\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}x\right) & \cosh\left(\frac{s}{a}x\right)\end{bmatrix}\cdot\begin{bmatrix}\{p(0,t)\}_{t\to s}\\ \{Q(0,t)\}_{t\to s}\end{bmatrix} -$$
$$\mathrm{H}(x-L)\begin{bmatrix}\cosh\left(\frac{s}{a}(x-L)\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}(x-L)\right)\\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(x-L)\right) & \cosh\left(\frac{s}{a}(x-L)\right)\end{bmatrix}\cdot\begin{bmatrix}b\{Q(L,t)\}_{t\to s}\\ sC\{p(L,t)\}_{t\to s}\end{bmatrix}. \quad (3.101)$$

In this case we have three unknown coefficients, $p(0,s), p(L,s), Q(L,s)$, hence we have to apply formally three operators defined in the Definition 7

$$\mathcal{E}_{2L} = \begin{bmatrix}0\\ \mathcal{E}_{2L}\end{bmatrix}, \text{ and } \mathcal{E}_L = \begin{bmatrix}\mathcal{E}_L\\ \mathcal{E}_L\end{bmatrix}.$$

Simplification of the system of equations (3.101) after we applied mentioned operators yields the matrix equation

$$\begin{bmatrix}-\frac{S}{a\rho}\sinh\left(\frac{2L}{a}s\right) & -cs\cosh\left(\frac{Ls}{a}\right) & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right)\\ -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & -cs & -1\\ \cosh\left(\frac{Ls}{a}\right) & -1 & -b\end{bmatrix}\cdot\begin{bmatrix}p(0,s)\\ p(L,s)\\ Q(L,s)\end{bmatrix} = \begin{bmatrix}-r\cosh\left(\frac{2L}{a}s\right)\\ -r\cosh\left(\frac{Ls}{a}\right)\\ \frac{ar}{S}\rho\sinh\left(\frac{Ls}{a}\right)\end{bmatrix} \quad (3.102)$$

Solution (3.103) of the matrix equation (3.102) gives us the vector of unknowns, which allows us to express the state vector $\mathbf{u}(x,s), \forall x \in \langle 0, 2L\rangle$

$$\begin{bmatrix}\frac{ar\rho\left(S^2b\sinh\left(\frac{2L}{a}s\right)+2Sa\rho\cosh\left(\frac{2L}{a}s\right)+a^2c\rho^2s\sinh\left(\frac{2L}{a}s\right)\right)}{S\left(S^2b\cosh\left(\frac{2L}{a}s\right)-S^2b+2Sa\rho\sinh\left(\frac{2L}{a}s\right)+a^2c\rho^2s\cosh\left(\frac{2L}{a}s\right)+a^2c\rho^2s\right)}\\ \frac{a^2r\rho^2\cosh\left(\frac{Ls}{a}\right)}{\frac{S^2b}{2}\cosh\left(\frac{2L}{a}s\right)-\frac{S^2b}{2}+Sa\rho\sinh\left(\frac{2L}{a}s\right)+\frac{cs}{2}a^2\rho^2\cosh\left(\frac{2L}{a}s\right)+\frac{cs}{2}a^2\rho^2}\\ \frac{Sar\rho\sinh\left(\frac{Ls}{a}\right)}{S^2b\cosh^2\left(\frac{Ls}{a}\right)-S^2b+2Sa\rho\sinh\left(\frac{Ls}{a}\right)\cosh\left(\frac{Ls}{a}\right)+a^2c\rho^2s\cosh^2\left(\frac{Ls}{a}\right)}\end{bmatrix}. \quad (3.103)$$

3.3.4 Piping system with ***parallel hydraulic resistance***

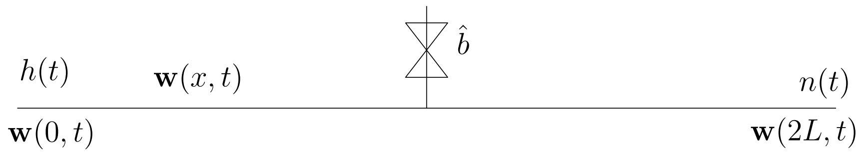


Fig. 3.14: Simple piping system, with *parallel hydraulic resistance.*

This problem was described in 3.2.7 and we can write down the system of equations for this case as

$$\frac{\partial p(x,t)}{\partial x} + \frac{\rho}{S}\frac{\partial Q(x,t)}{\partial t} = 0$$
$$\frac{\partial Q(x,t)}{\partial x} + \frac{S}{\rho a^2}\frac{\partial p(x,t)}{\partial t} + \frac{1}{\hat{b}}\,p(\zeta,t)\,\delta(x-\zeta) = 0,$$

which results in the problem of finding two unknown coefficients. The vector of those parameters is the solution of the matrix equation

$$\begin{bmatrix} -\frac{S}{a\rho}\sinh\left(\frac{2L}{a}s\right) & -\frac{1}{\hat{b}}\cosh\left(\frac{Ls}{a}\right) \\ \cosh\left(\frac{Ls}{a}\right) & -1 \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{p(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -r\cosh\left(\frac{2L}{a}s\right) \\ \frac{ar}{S}\rho\sinh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.104}$$

The solution gives us the tools to express the *Laplace image* of the state vector $\mathbf{u}(x,s)$, $\forall x \in \langle 0, 2L\rangle$.

3.3.5 Piping system with *centrifugal pump*

As we saw in the subsection 3.2.8, the piping system with the *centrifugal pump* is the most complicated, since none of the coefficients b, C, σ, κ in (3.89) is cancelled. results in the matrix equation (3.106)

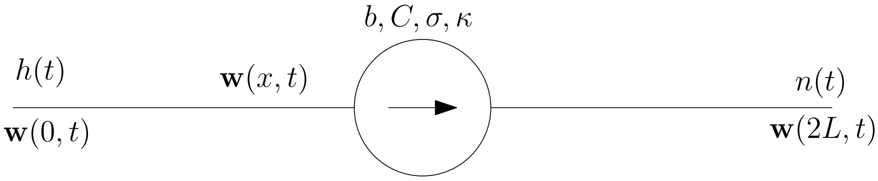


Fig. 3.15: Simple piping system, with *centrifugal pump*.

Substituting $\zeta = L$ and formally applying two operators from the Definition 7 gives the equations

$$\begin{bmatrix} \{Q(2L,t)\}_{t\to s} \\ \{p(L,t)\}_{t\to s} \\ \{Q(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -\frac{S}{\rho a}\sinh\left(\frac{s}{a}2L\right) & \cosh\left(\frac{s}{a}2L\right) \\ \cosh\left(\frac{s}{a}L\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}L\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}L\right) & \cosh\left(\frac{s}{a}L\right) \end{bmatrix} \cdot \begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{Q(0,t)\}_{t\to s} \end{bmatrix} +$$
$$\mathrm{H}(x-\zeta) \begin{bmatrix} -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(2L-L)\right) & \cosh\left(\frac{s}{a}(2L-\zeta)\right) \\ \cosh\left(\frac{s}{a}(L-L)\right) & \frac{-\rho a}{S}\sinh\left(\frac{s}{a}(L-L)\right) \\ -\frac{S}{\rho a}\sinh\left(\frac{s}{a}(L-L)\right) & \cosh\left(\frac{s}{a}(L-L)\right) \end{bmatrix} \cdot \begin{bmatrix} \sigma\{p(\zeta,t)\}_{t\to s} - b\{Q(\zeta,t)\}_{t\to s} \\ -sC\{p(\zeta,t)\}_{t\to s} - s\kappa\{Q(\zeta,t)\}_{t\to s} \end{bmatrix}, \tag{3.105}$$

thus we can rewrite the equation (3.105) into the

$$\begin{bmatrix} -\frac{S}{a\rho}\sinh\left(\frac{2L}{a}s\right) & -\frac{S\sigma}{a\rho}\sinh\left(\frac{Ls}{a}\right) - cs\cosh\left(\frac{Ls}{a}\right) & \frac{Sb}{a\rho}\sinh\left(\frac{Ls}{a}\right) - \kappa s\cosh\left(\frac{Ls}{a}\right) \\ \cosh\left(\frac{Ls}{a}\right) & \sigma - 1 & -b \\ -\frac{S}{a\rho}\sinh\left(\frac{Ls}{a}\right) & -cs & -\kappa s - 1 \end{bmatrix} \cdot$$
$$\begin{bmatrix} \{p(0,t)\}_{t\to s} \\ \{p(L,t)\}_{t\to s} \\ \{Q(L,t)\}_{t\to s} \end{bmatrix} = \begin{bmatrix} -r\cosh\left(\frac{2L}{a}s\right) \\ \frac{afce}{S}\rho\sinh\left(\frac{Ls}{a}\right) \\ -r\cosh\left(\frac{Ls}{a}\right) \end{bmatrix}. \tag{3.106}$$

Solving the matrix equation (3.106) gives the unknown coefficients. As we substitute these coefficients into (3.92) we get the *Laplace images* of the pressure and flow in the considered piping system.

4 CONSLUSION

This work aimed to present possibilities of utilizing generalized functions for non-stationary problems in continuum mechanics.

Chapter 1 contained building the theory of distributions, as continuous linear functionals on the space of the test functions. This was necessary to understand the singular distribution known as the *Dirac delta function*, which advantages of usage were presented in the following chapters. The second part of the theoretical framework included the *Laplace transform* integral method, which was used to reduce partial differential equations into the differential ones. It was necessary to implement double *Laplace transform* according to the time variable and a spatial one because we studied one-dimensional non-stationary problems in this work.

The following Chapter 2 dealt with the utilization of the generalized functions in the transverse vibrations of continuous beams. Section 2.1 covered a derivation of the dynamic beam equation according to the Euler-Bernoulli theory. Further, the solution of the beam's deflection under the point non-stationary loads, which were modelled using *Dirac delta function*, was shown. The analysis of natural frequencies in subsection 2.2.2 compares Euler-Bernoulli's analytical solution with the numerical one from ANSYS[1], under a student license. This comparison showed that the simplification of the linear theory of elasticity gives higher values of natural frequencies and more significant difference can be observed with the increasing mode shape.

The example in the section 2.3, presented modelling of different ends of the beam: clamp, simple support and the free end are approximated using torsional and translational springs. These modelling options form the basis for creating a general beam solution model mounted on the n-flexible supports described in the section 2.4. The above-mentioned general model is the principle of the algorithm in the attached software for solving the beam deflection.

At the beginning of the Chapter 3, dealing with hydrodynamics, the equations (3.47) and (3.48) describing non-stationary fluid flow in pipes were derived. Equations of continuity and equilibrium form a system of partial differential equations and describe the development of pressure and flow in the pipeline. Description and solution of the one-dimensional problem in a simple tube, using double *Laplace transform*, was presented. The opportunities that offer generalized functions are apparent in the piping system that is set up from multiple pipes connected in series using the hydraulic elements listed in the Table 3.1.

The solution of such systems using distributions allows working with the original system of equations for the entire piping system. These hydraulic elements are included in the calculation using the *Dirac delta function*. The classical calculation method presented in [9] solves the problem of the piping system by dividing it into individual sections. At the end of this thesis, the methods of calculation by both methods for all hydraulic elements from the table were presented.

The work of this thesis can be further extended to the 2-dimensional problems in the Chapter 2 and into the branch piping systems in the last chapter.

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LIST OF SYMBOLS, PHYSICAL CONSTANTS AND ABBREVIATIONS

Symbol	Unit	Meaning
α	[-]	*Multi-index*
$C^\infty(\Omega)$	[-]	Smooth functions defined on Ω
$C_0^\infty(\Omega)$	[-]	Smooth functions with compact support on Ω
$\mathscr{D}(\Omega)$	[-]	Space of the test functions defined on Ω
$\mathscr{D}^*(\Omega), \mathscr{D}'(\Omega)$	[-]	*Algebraic, topological dual space*
$\delta_a = \delta(x-a)$	[-]	*Dirac delta function* defined in a
ε	[-]	Parameter of the *Laplace transform* $\mathscr{L}_{x\to\varepsilon}$
$\mathcal{E}$	[-]	*multi-point* differential operator
$H_a = H(x-a)$	[-]	*Heaviside step function* defined in a
K	[-]	Compact set
$\mathrm{L}^1_{loc}(\Omega)$	[-]	Locally integrable functions in Ω
$\mathscr{L}$	[-]	*Laplace transform*
$\mathscr{L}^{-1}$	[-]	Inverse *Laplace transform*
Ω	[-]	Open connected set
$\mathbb{R}, \mathbb{C}$	[-]	The set of real, complex numbers
s	[-]	Parameter of the *Laplace transform* $\mathscr{L}_{t\to s}$
$S(\lambda x), T(\lambda x)$	[-]	*Rayleigh functions*
$U(\lambda x), V(\lambda x)$	[-]	*Rayleigh functions*
$\langle T, \varphi\rangle$	[-]	The value of the distribution T on the $\varphi \in \mathscr{D}$
$*$	[-]	Convolution sign
A	[$\mathrm{m^2}$]	Area of the cross-section
c_e	[$\mathrm{N\,m^{-2}}$]	Flexible subsoil coefficient
E	[Pa]	Young's modulus
$F(t)$	[N]	Non-stationary force
I	[$\mathrm{m^4}$]	Area moment of inertia
$\mathbb{I}$	[-]	Identity matrix of dimension 2
k_e	[$\mathrm{N\,s\,m^{-2}}$]	External damping coefficient
k_o	[N m]	Stiffness of the torsional spring
k_p	[$\mathrm{N\,m^{-1}}$]	Stiffness of the translational spring coefficient
L	[m]	Length of the beam
$M(t)$	[N m]	Non-stationary moment
Ω_j	[$\mathrm{rad\,s^{-1}}$], [Hz]	j-th natural frequency
Ω	[$\mathrm{rad\,s^{-1}}$], [Hz]	Frequency of the external load
ρ	[$\mathrm{kg\,m^{-3}}$]	Density
t	[s]	Time
$y(x,t)$	[m]	Beam deflection in the position x at the time t
$\mathbf{0}$	[-]	Zero matrix of dimension 2
a	[$\mathrm{m\,s^{-1}}$]	Speed of the pressure wave in compressible fluid
$\alpha(t)$	[-]	Unknown function at the left end of the pipe
$\mathbf{a_1}, \mathbf{a_2}, \mathbf{b_1}, \mathbf{b_2}$	[-]	Boundary conditions vectors
b	[$\mathrm{kg\,m^{-4}\,s^{-1}}$]	Characteristic of the *hydraulic resistance*

Symbol	**Unit**	**Meaning**
$\hat{b}$	$[\mathrm{kg\,m^{-4}\,s^{-1}}]$	Characteristic of the *parallel hydraulic resistance*
$\beta(s)$	[-]	*Laplace transform* $\mathscr{L}\{\alpha(t)\}_{t\to s}$
$\mathbf{c}, c_i$	$[\mathrm{m\,s^{-1}}]$	Flow velocity
C	$[\mathrm{m^4\,s^2\,kg^{-1}}]$	Characteristic of the *gas accumulator*
$\mathcal{C}$	[-]	Unknown constant of integration
$\frac{\mathrm{D}}{\mathrm{D}t}$	[-]	Substantial derivative
ϵ	[-]	Fractional tension
E_k	$[\mathrm{kg\,m^{-1}\,s^{-2}}]$	Fluid elastic modulus
$\mathbf{g}, g_i$	$[\mathrm{m\,s^{-2}}]$	Gravitational acceleration
K	$[\mathrm{kg\,m^{-1}\,s^{-2}}]$	-
$\mathbf{m}(t)$	[-]	Unknown function at the right end of the pipe
M	[-]	Mach number
$\mathbf{n}$	[-]	Outward pointing unit normal
$n(t)$	[-]	Known function at the right end of the pipe
$p(x,t)$	[Pa]	Pressure
π_{ij}	[Pa]	Tensor of viscous stress
p_0, Q_0	[Pa], $[\mathrm{m^3\,s^{-1}}]$	Stationary part of pressure, flow
$\mathbf{P_i}$	[-]	Transfer matrix of the pipe i
$Q(x,t)$	$[\mathrm{m^3\,s^{-1}}]$	Flow rate
$\mathbf{r}(0,s)$	[-]	*Laplace transform* $\mathscr{L}\{\mathbf{m}(0,t)\}_{t\to s}$
R, D	[m]	Pipe radius, diameter
$\mathbf{R_i}$	[-]	Transient matrix of the pipe i
σ, q	[Pa], $[\mathrm{m^3\,s^{-1}}]$	Non-tationary part of pressure, flow
σ, κ	[-]	Cavitation coefficients of the centrifugal pump
σ_{ij}, Σ_{ij}	[Pa]	Tensile stress, Pipe tensile stress
$S(x,t)$	$[\mathrm{m^2}]$	Area of the cross section of the pipe
$\mathbf{u}(x,s)$	[-]	*Laplace transform* $\mathscr{L}\{\mathbf{w}(x,t)\}_{t\to s}$
$V(t)$	$[\mathrm{m^3}]$	Volume at the time t
v_{ij}	$[\mathrm{s^{-1}}]$	Speed deformation tensor
$\mathbf{w}(x,t)$	[-]	Vector of pressure $p(x,t)$ and flow $Q(x,t)$
w_t	[s]	Thickness of the wall of the pipe
$z(s)$	[-]	*Laplace transform* $\mathscr{L}\{n(t)\}_{t\to s}$

CONTENTS OF THE ATTACHED CD

```
/..........................................................root of the CD
├── software
│   ├── non-stationary..........................folder with non-stationary python files
│   │   ├── nstatM_proch.py
│   │   ├── nstatMC_proch.py
│   │   ├── maple_print.py
│   │   └── nstat_proch.txt
│   └── stationary..............................folder with non-stationary python files
│       ├── stat_proch.py
│       └── stat_proch.txt
└── pdf
    └── 2018_DP_Prochazka_Petr_160685.pdf.............electronic version of this work
```